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ADMINISTRATIVE INFORMATION

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1.0 BACKGROUND

1.1 Introduction

In integrated Global Positioning System/Inertial Navigation System (GPS/INS) navigation systems, position and velocity errors are typically estimated with respect to some reference point within the host vehicle. For example, this may be the point afforded by the origin of the accelerometer triad or the center of mass of the host vehicle. The conventional approach is to relate measurement residuals to error states under the assumption that the antenna phase center coincides with the reference point while compensating for the effects of any lever arm.

In reality, the antenna phase center is usually at a point different from the reference point. Measurement residuals (observations) formed by differencing predicted and measured pseudo range and delta range must incorporate corrections for the antenna-phase-center lever arm. Knowing the placement of the GPS antenna with respect to the host vehicle and the host-vehicle attitude with respect to some convenient reference frame provides the required information. But, as the conventional observation matrix is based on coincidence of the antenna phase center and the reference point (notwithstanding that in reality it may be offset), the pseudo-range and delta-range measurement residuals are modeled to have coupling only to position, velocity, user clock bias and drift error states. In this conventional approach, potential coupling to attitude errors or attitude rate errors is ignored.

In the conventional approach, the needed attitude for antenna-lever-arm correction is generally provided by estimation of attitude errors and propagation of the corrected attitude "total state" (typically a direction cosine matrix or quaternion) to the time(s) of measurement. Attitude-error estimation for such an implementation depends entirely on correlation between the attitude error and the velocity error. Specifically, contribution of the attitude error to the velocity error is due to the product of the attitude-error vector by the skew-symmetric matrix of specific forces within the dynamics F matrix.

This approach has been successfully applied to many applications such as terrestrial navigation. There are, however, applications where the correlation between attitude error and velocity error vanishes. For example, when an exoatmospheric host vehicle is free-falling, the specific forces are zero for all practical purposes. Then, although GPS may succeed in estimating position and velocity errors fairly well, attitude estimation is not possible. Moreover, the correction for the antenna lever arm becomes flawed as attitude errors grow due to instrumentation errors such as gyro bias and scale factor errors.

Typically, GPS-based attitude error estimation methods rely on receiving carriers from multiple GPS satellites and processing the single difference of their phases (Van Graas & Braasch, 1991-2; Keirleber & Maki, 1991; Satz et al., 1991). A minimum of 3 noncollinear antennas are required in most methods. The implementation of these methods, however, requires considerable receiver hardware design features and/or special software processing beyond the typical GPS/INS designs.

An alternative approach to attitude estimation is proposed in this report. We seek to exploit the attitude information inherently present in the pseudo-range and the delta-range measurements obtained with a single offset GPS antenna mounted on a platform undergoing attitude changes. The attitude information is recovered with standard recursive estimation techniques such as the Kalman Filter algorithms of the GPS/INS systems. The attitude information is primarily recovered through processing delta-range measurements. The pseudo-range measurement is less useful because of the generally high code loop noise. The implementation of this approach does not require any special receiver hardware and only a few minor changes to the standard GPS/INS Kalman Filter software. Specifically, the required changes to the Kalman Filter software are simply a few additional entries in the observation matrix H.

The proposed approach is ideally suited for applications where the antenna is mounted on a spinning platform such as a spinning vehicle in exoatmospheric, free-fall conditions. Since this approach imposes a minor implementation cost, it readily allows use of existing systems. For "good" performance, an integrated GPS/INS system is required; the INS provides for propagation of the attitude estimates in between GPS updates. In some specific applications, however, GPS alone may suffice.

An overview of the contents of this report is as follows. Sections 2, 3, and 4 contain independent mathematical developments of this approach to attitude estimation. Section 2 provides a general mathematical development of the observation matrix H, which is applicable to any changing host vehicle attitude. The H matrix is developed without reference to any of the other models that are part of a GPS/INS Kalman Filter. Section 2 also provides a quick comparison of this method to the more familiar "interferometric" GPS-based attitude determination methods. Section 3 provides a mathematical development of the observation matrix H, which is also generally applicable to any changing host-vehicle attitude. This development, however, is based on a state-space formulation and as a consequence it employs and thereby relies on prior developments - specifically, the state transition matrix and the plant-noise model. Section 4 provides a simpler analysis of attitude estimation but limited to a spinning body scenario. As a result it facilitates gaining certain insights to and identifying sensitivities in spinning body applications. Section 5 provides simulation results for a spinning-body application by using,

the more general formulation of Section 3. Section 6 provides a summary and conclusions.

1.2 Notation

Vectors in specific coordinate frames will generally be of lower case and always with a single underline. Coordinate free forms will be denoted with an over-bar. Matrices will generally be of upper case and always with a double underline. Scalars may be upper or lower case but without any underline. Normal superscript designation of the reference frame will be suppressed when the frame is the preferred frame. The subscript "n" will be appended to terms for the nominal trajectory.

2.0 GENERAL DERIVATION OF THE OBSERVATION MATRIX H

2.1 Introduction

This section provides a general mathematical development of the observation matrix H, which is applicable to any changing host-vehicle attitude. The pseudo-range (PR) and delta-range (DR) measurement residuals obtained from the corresponding measurements and their prediction at the antenna phase center are related to estimation quantities referred to the reference point. This is done to capture potential attitude and attitude rate information lost in the more typical process of computationally bringing the antenna phase center to the reference point (but using lever-arm corrections in formation of the residuals). If coupling to attitude exists, then the attitude of a host vehicle undergoing attitude changes as well its position and velocity can be estimated even if the specific forces are zero.

The following analysis first derives H matrix row vectors corresponding to pseudo-range residuals for a single "offset" GPS antenna and then proceeds to derive H matrix row vectors for delta range residuals. GPS pseudo-range measurement performance is based on the code tracking loop, while that for the delta range is based on the corresponding time increments of the carrier loop. It is recognized that for limitations in the antenna-lever-arm length, the code loop noise will generally overwhelm the relation of pseudo-range residuals to attitude error. The delta range residuals, however, being based on the very-low-noise carrier tracking may well afford estimation of attitude errors of the free falling host vehicle (as well as provide estimation of attitude rate error so that gyro errors may be observed). The derived H matrix includes rows for both pseudo-range and delta-range residuals for completeness and improvement that may accrue from pseudo-range measurements over a period of time (reduction of the effect of code loop noise through integration).

The more general H matrix derivations are simply extended to multiple antennas but this is of little value to the problem of interest as it would entail considerable hardware complication with attendant cost. A brief investigation of higher order terms in the temporal DR residuals process is included.

A rotating antenna by itself introduces (by virtue of its rotation) carrier phase shifts beyond those associated with the DR measurements including a fundamental phase change due to the antenna rotation ($\pm$ 360 degrees of phase shift per 360 degrees of rotation in the antenna plane) and phase anomalies peculiar to the antenna phase pattern. These phase shifts must be identified and compensation made. It is assumed that the INS can measure the antenna rotation with sufficient accuracy so that the unwanted phase shifts as well as other antenna phase imperfections can be compensated. The problems and issues associated with the

corresponding compensation of delta-range residuals and antenna calibrations were not investigated here.

2.2 Pseudo-Range Measurement

The geometry of the problem is shown in figure 1. The reader should be aware of the direction taken for the unit vector, which is shown from the jth GPS satellite to the phase center of the offset GPS antenna on the host vehicle. This may be of opposite sense to some definitions used by others. This unit vector is also well approximated by a unit vector from the jth satellite toward the user as the length of the antenna lever arm is very small in comparison to the range to the satellite. Likewise, in this analysis, the sign of error states is such that when added to the predicted (propagated) "nominal" total state, a new estimation of the total state is provided (which becomes the new nominal total state in an extended Kalman filter). The pseudo range between the antenna phase center and the jth satellite is given by

$$z_j = (\overline{r}_u + \overline{r}_A - \overline{r}_{sj}) \cdot \overline{e}_j + B_u \, , \tag{1}$$

where $\overline{r}_u, \overline{r}_A$, and $\overline{r}_{sj}$ represent coordinate free vector form. As bias in the satellite clock is not modeled, it is not included in (1). In (2) we have coordinatized with respect to some convenient reference frame Σ_a. Frame notations are such that

Σ_e - Earth frame, generally Earth Centered Earth Fixed (ECEF)

Σ_i - Inertial frame, generally Earth Centered Inertial (ECI)

Σ_n - Local level frame

Σ_b - Body (host vehicle) frame.

As a random process, the pseudo range is given by

$$z_j = \underline{e}_j^{aT} (\underline{r}_u^a + \underline{r}_A^a - \underline{r}_{sj}^a) + B_u + v_{PRj}$$

or

$$z_j = \underline{e}_j^T (\underline{r}_u + \underline{r}_A - \underline{r}_{sj}) + B_u + v_{PRj} \tag{2}$$

when suppressing the superscript "a" for notational clarity. We will frequently suppress the superscript "a" for our selected convenient reference frame but maintain superscripts for other specially designated frames.

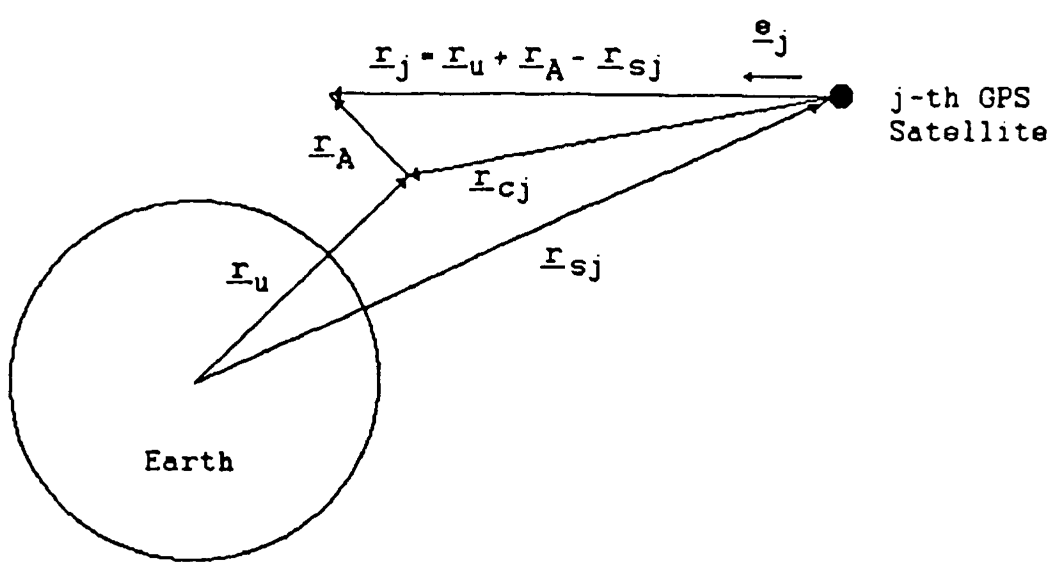


Figure 1. Problem geometry.

$\underline{r}_A$ is the random process vector providing the location of the GPS antenna but as represented in Σ_a. $\underline{r}_A$ will be rapidly changing due to the changing attitude of the host vehicle. However, when represented in the host vehicle frame (as $\underline{r}^b{}_A$), it will be constant and related by

$$\underline{r}_A(t) = \underline{\underline{C}}_b^a(t)\,\underline{r}_A^b \ . \tag{3}$$

It is to be noted that the direction cosine matrix $\underline{\underline{C}}_b^a$ is modeled as a random process but not $\underline{r}^b{}_A$, which is taken to be well known and constant.

For purposes of the derivation of the H matrix for an extended (or linearized) Kalman filter, we assume a continuous spatial as well as temporal distribution of z_j, $\underline{r}_u$, and $\underline{r}_A$. In the approach of this section it is convenient to disregard pseudo-range and delta-range measurement noise. The spatial differential which we will take with respect to the nominal or predicted value (i.e. evaluated along the nominal trajectory), corresponds to the pseudo range measurement residual. That is, $\delta(\cdot) = (\cdot) - (\cdot)_n$ where we note

$$\begin{aligned}\delta(\underline{p}^T\underline{q}) &= \underline{p}^T\underline{q} - \underline{p}_n^T\underline{q}_n = (\underline{p}_n + \delta\underline{p})^T(\underline{q}_n + \delta\underline{q}) - \underline{p}_n^T\underline{q}_n \\ &\approx (\delta\underline{p})^T\underline{q}_n + \underline{p}_n^T\delta\underline{q} = \underline{q}_n^T\delta\underline{p} + \underline{p}_n^T\delta\underline{q}\ .\end{aligned}$$

In particular,

$$\begin{aligned}\delta z_j &= \delta[\underline{e}_j^T(\underline{r}_u + \underline{r}_A - \underline{r}_{sj})] + \delta B_u \\ \delta z_j &= (\underline{r}_{un} + \underline{r}_{An} - \underline{r}_{sjn})^T\delta\underline{e}_j + \underline{e}_{jn}^T\delta(\underline{r}_u + \underline{r}_A - \underline{r}_{sj}) + \delta B_u\ .\end{aligned} \tag{4}$$

As $\underline{e}_j$ is of constant (unit) magnitude, $\delta\underline{e}_j$ for small magnitudes must be orthogonal to $\underline{e}_j$ and, therefore, to $\underline{r}_u + \underline{r}_A - \underline{r}_{sj}$ to which $\underline{e}_j$ is parallel. Hence,

$$\begin{aligned}\delta z_j &= \underline{e}_{jn}^T\,\delta\underline{r}_u + \underline{e}_{jn}^T\,\delta\underline{r}_A + \delta B_u \\ \delta z_j &= \underline{e}_{jn}^T\,\delta\underline{r}_u + \underline{e}_{jn}^T(\delta\underline{\underline{C}}_b^a)\,\underline{r}_{An}^{bn} + \delta B_u\end{aligned} \tag{5}$$

as $\underline{r}_{An}^{bn} = \underline{r}_A^b$ is a constant (column) vector.

In the case of $\delta\underline{\underline{C}}^{a}_{b}$ where the spatial variability is due to that of Σ_{b},

$$\delta\underline{\underline{C}}^{a}_{b} = \underline{\underline{C}}^{a}_{b} - \underline{\underline{C}}^{a}_{bn} = \underline{\underline{C}}^{a}_{bn}\,(\underline{\underline{C}}^{bn}_{b} - \underline{\underline{I}}) \ . \tag{6}$$

For small displacements of Σ_{b} from Σ_{bn} (the nominal body frame), $\underline{\underline{C}}^{bn}_{b} - \underline{\underline{I}}$ is skew-symmetric. We define

$$\underline{\underline{\delta\Psi}}^{bn} = \underline{\underline{\delta\Psi}}^{bn}_{bn,b} = \underline{\underline{C}}^{bn}_{b} - \underline{\underline{I}} \ . \tag{7}$$

For notational convenience, δ is used in this definition but this does not allow for separation of δ from $\underline{\underline{\delta\Psi}}$ as this would presume that $\underline{\underline{\Psi}}$ has meaning for large displacements of Σ_{b} from Σ_{bn}. $\underline{\underline{\delta\Psi}}$ is not defined. We then have

$$\delta\underline{\underline{C}}^{a}_{b} = \underline{\underline{C}}^{a}_{bn}\,\underline{\underline{\delta\Psi}}^{bn} \ . \tag{8}$$

Therefore,

$$\delta z_{j} = \underline{e}^{T}_{jn}\,\delta\underline{r}_{u} + \underline{e}^{T}_{jn}\,\underline{\underline{C}}^{a}_{b}\,\underline{\underline{\delta\Psi}}^{bn}\,\underline{r}^{bn}_{An} + \delta B_{u} \ . \tag{9}$$

But as

$$\underline{\underline{C}}^{a}_{bn}\,\underline{\underline{\delta\Psi}}^{bn}\,\underline{r}^{bn}_{An} = \underline{\underline{C}}^{a}_{bn}\,\underline{\underline{\delta\Psi}}^{bn}\,\underline{\underline{C}}^{bn}_{a}\,\underline{\underline{C}}^{a}_{bn}\,\underline{r}^{bn}_{An} = \underline{\underline{\delta\Psi}}\,\underline{r}_{An} \ , \tag{10}$$

and since

$$\underline{\underline{\delta\Psi}}\,\underline{r}_{An} = -\underline{\underline{R}}_{An}\,\underline{\delta\phi}, \qquad \underline{\underline{R}}_{An} = \begin{bmatrix} 0 & -r_{Azn} & r_{Ayn} \\ r_{Azn} & 0 & -r_{Axn} \\ -r_{Ayn} & r_{Axn} & 0 \end{bmatrix} , \tag{11}$$

we have

$$\delta z_{j} = \underline{e}^{T}_{jn}\,\delta\underline{r}_{u} - \underline{e}^{T}_{jn}\,\underline{\underline{R}}_{An}\,\underline{\delta\phi} + \delta B_{u} \ . \tag{12}$$

$\underline{\underline{R}}_{An}$ is the skew-symmetric matrix form of vector $\underline{r}_{An}$ and $\underline{\delta\phi}$ is the vector form of skew-symmetric $\underline{\underline{\delta\Psi}}$. Note that δ cannot be separated from $\underline{\delta\phi}$ in the context used.

The H matrix for pseudo-range measurements only but referred to the GPS antenna phase center is given by[1]

$$
\begin{array}{c} \\ \delta z_1 \\ \delta z_2 \\ \vdots \\ \delta z_k \end{array}
\begin{array}{c}
\begin{array}{ccc} \delta \underline{r}_u & \delta \underline{\phi} & \delta B_u \end{array} \\
\begin{bmatrix}
\underline{e}_{1n}^T & -\underline{e}_{1n}^T \underline{\underline{R}}_{An} & 1 \\
\underline{e}_{2n}^T & -\underline{e}_{2n}^T \underline{\underline{R}}_{An} & 1 \\
\vdots & \vdots & \vdots \\
\underline{e}_{kn}^T & -\underline{e}_{kn}^T \underline{\underline{R}}_{An} & 1
\end{bmatrix}
\end{array} . \qquad (13)
$$

If, instead of bringing the H matrix to the antenna phase center, had we derived an H matrix based on the antenna phase center coinciding with the reference point, that is $\underline{\underline{R}}_A = \underline{\underline{O}}$ (but still using measurement residuals based on antenna-lever-arm corrections), the resulting H matrix would show no coupling between the residuals and $\underline{\delta\phi}$. By eliminating this noncoupled term, we have

$$
\begin{array}{c} \\ \delta z_1 \\ \delta z_2 \\ \vdots \\ \delta z_k \end{array}
\begin{array}{c}
\begin{array}{cc} \delta \underline{r}_u & \delta B_u \end{array} \\
\begin{bmatrix}
\underline{e}_{1n}^T & 1 \\
\underline{e}_{2n} & 1 \\
\vdots & \vdots \\
\underline{e}_{kn}^T & 1
\end{bmatrix}
\end{array} , \qquad (14)
$$

which is the usual H matrix encountered when pseudo-range measurements are presented to a Kalman filter implemented in ECEF.

The more general H matrix of (13) appears to have promise for observability for attitude but would require a large offset distance of the antenna phase center from the reference point of estimated quantities to overcome the noise in the GPS receiver code loops. Common sense dictates that, with a single antenna, regardless of the number of GPS satellites for which pseudo-range measurement are made at the same time, full determination would not be possible without prior information from other measurement times. The H matrix will not be of full rank for a single antenna scheme. In particular, three separate times of measurement with the offset antenna occupying three different noncollinear positions are required for full attitude observability, i.e., ensure full rank of (13). If we constrain the attitude variability of the free-falling host vehicle to spin about a known inertially stable axis, then

---

[1] Remember that suppression of superscripts for vectors applies only to Σ_a.

only two measurement times with noncoincident antenna positions should be required for full observability. With multiple antennas (not considered here), unique or overdetermination at a single measurement time is certainly possible.

2.3 Delta-Range Measurements

With respect to the pseudo-range measurement residuals, we have a first order approximation to a spatial difference process (which involves difference with respect to a nominal trajectory). Proceeding onward to include delta-range measurements, we now seek the spatial difference process of a temporal difference process. We will maintain a first order approximation to the spatial difference part in order to use a linear Kalman filter. The temporal difference will be initially given a first order approximation. Later, a second order approximation will be examined to see what additional terms (preferably involving existing error states) have coupling to the delta range measurements.

Taking the time differential of the pseudo-range measurement residual we have

$$d(\delta z_j) = d(\underline{e}^T_{jn}\,\delta\underline{r}_u) - d(\underline{e}^T_{jn}\,\underline{\underline{R}}_{An}\,\underline{\delta\phi}) + d(\delta B_u), \tag{15}$$

and on expanding,

$$\begin{aligned} d(\delta z_j) = {} & (d\underline{e}^T_{jn})\,\delta\underline{r}_u + \underline{e}^T_{jn}\,d(\delta\underline{r}_u) \\ & -(d\underline{e}^T_{jn})\,\underline{\underline{R}}_{An}\,\underline{\delta\phi} - \underline{e}^T_{jn}\,(d\underline{\underline{R}}_{An})\,\underline{\delta\phi} - \underline{e}^T_{jn}\,\underline{\underline{R}}_{An}\,d(\underline{\delta\phi}) + d(\delta B_u). \end{aligned} \tag{16}$$

By linearity we can exchange $d(\cdot)$ and $\delta(\cdot)$ for all except $\underline{\delta\phi}$, which, as previously noted, is not separable with any proper meaning. For $\underline{\delta\phi}$ we must return to the defining relations. From (7) and (8), on considering the temporal behavior we have

$$\begin{aligned} \delta\underline{\underline{C}}^a_b &= \underline{\underline{C}}^a_{bn}\,\underline{\underline{\delta\Psi}}^{bn}, \quad \underline{\underline{\delta\Psi}}^{bn} = \underline{\underline{C}}^{bn}_b - \underline{\underline{I}} \\ \underline{\underline{\delta\Psi}} \triangleq \underline{\underline{\delta\Psi}}^a &= \underline{\underline{C}}^a_{bn}\,(\underline{\underline{C}}^{bn}_b - \underline{\underline{I}})\,\underline{\underline{C}}^{bn}_a = \underline{\underline{C}}^a_b\,\underline{\underline{C}}^{bn}_a - \underline{\underline{I}} \\ D(\underline{\underline{\delta\Psi}}) &= (D\underline{\underline{C}}^a_b)\,\underline{\underline{C}}^{bn}_a + \underline{\underline{C}}^a_b\,(D\underline{\underline{C}}^{bn}_a) \approx (D\underline{\underline{C}}^a_b)\,\underline{\underline{C}}^b_a + \underline{\underline{C}}^a_{bn}\,(D\underline{\underline{C}}^{bn}_a) \\ D(\underline{\underline{\delta\Psi}}) &= \underline{\underline{\Omega}}^a_{ab} + \underline{\underline{\Omega}}^a_{bn,a} = \underline{\underline{\Omega}}^a_{ab} - \underline{\underline{\Omega}}^a_{a,bn} = \underline{\underline{\Omega}}_{ab} - \underline{\underline{\Omega}}_{a,bn} = \delta\underline{\underline{\Omega}}_{ab}\,. \end{aligned} \tag{17}$$

$\underline{\underline{\Omega}}_{ab}$ is the skew-symmetric matrix form of angular velocity of the host-vehicle frame with respect to the selected convenient reference frame. The corresponding vector form is

$$
\begin{aligned}
D(\underline{\delta\phi}) &= \underline{\omega}_{ab} - \underline{\omega}_{a,bn} = \delta\underline{\omega}_{ab} \\
d(\underline{\delta\phi}) &= \delta\underline{\omega}_{ab} dt ,
\end{aligned}
\tag{18}
$$

which is the mathematical statement of the fact that the rate of attitude error is equal to the angular velocity error.

It is to be noted here that δ is separated from $\underline{\omega}_{ab}$, as angular velocity is a vector quantity without further qualification. The relations of (18) are intuitively obvious but we have been careful to preserve formality. Since our focus will be on resolution of $\delta\underline{\omega}_{ab}$ into gyro error states[2] it is noted that if we take Σ_a to be the actual frame and not "what we think it is" irrespective of our selection for Σ_a, then

$$
\begin{aligned}
\delta\underline{\omega}_{ab} &= \underline{\omega}_{ab} - \underline{\omega}_{a,bn} = (\underline{\omega}_{ai} + \underline{\omega}_{ib}) - (\underline{\omega}_{ai} + \underline{\omega}_{i,bn}) \\
\delta\underline{\omega}_{ab} &= \underline{\omega}_{ib} - \underline{\omega}_{i,bn} = \delta\underline{\omega}_{ib} .
\end{aligned}
\tag{19}
$$

From (16) and (19) we now have

$$
\begin{aligned}
\delta(dz_j) = {} & (d\underline{e}_{jn}^T)\, \delta\underline{r}_u + \underline{e}_{jn}^T (\delta\underline{v}_u)\, dt \\
& - (d\underline{e}_{jn}^T)\, \underline{\underline{R}}_{An} \underline{\delta\phi} - \underline{e}_{jn}^T (d\underline{\underline{R}}_{An}) \underline{\delta\phi} \\
& - \underline{e}_{jn}^T \underline{\underline{R}}_{An} (\delta\underline{\omega}_{ib})\, dt + \delta f_u dt ,
\end{aligned}
\tag{20}
$$

where we have used $d\underline{r}_u = \dfrac{\partial \underline{r}_u}{\partial t} dt = \underline{v}_u dt$ and $dB_u = \dfrac{\partial B_u}{\partial t} dt - f_u dt$.[3]

---

[2] Resolution into gyro error states does not imply that these error states will be observable.

[3] $\underline{v}_u$ is the velocity of the user relative to the center of Earth as determined by a Σ_a fixed observer. $\underline{v}_u$ is denoted here as terrestrial velocity when $\Sigma_a = \Sigma_e$, and $\underline{v}_u$ is denoted as inertial velocity when $\Sigma_a = \Sigma_i$. For $\Sigma_a = \Sigma_n$, a local level frame, $\underline{v}_u$ is much less useful. We proceed, however, along this path and make adjustments later for expression in terms of terrestrial velocity regardless of the selection for Σ_a.

Using the Theorem of Mean Value for integration and taking $\delta\underline{r}_u$ and $\underline{\delta\phi}$ to be constant over the interval (t_i-T, t_i), the DR measurement residual is given by

$$\int_{t_i-T}^{t_i}\delta(dz_j) = \delta\int_{t_i-T}^{t_i}dz_j = \left(\int_{t_i-T}^{t_i}d\underline{e}_{jn}^T\right)\delta\underline{r}_u + \underline{e}_{jn}^T\int_{t_i-T}^{t_i}\delta\underline{v}_u dt$$

$$-\left(\int_{t_i-T}^{t_i}d\underline{e}_{jn}^T\right)\underline{\underline{R}}_{An}\underline{\delta\phi} - \underline{e}_{jn}^T\left(\int_{t_i-T}^{t_i}d\underline{\underline{R}}_{An}\right)\underline{\delta\phi} \tag{21}$$

$$-\underline{e}_{jn}^T\underline{\underline{R}}_{An}\int_{t_i-T}^{t_i}\delta\underline{\omega}_{ib}dt + \int_{t_i-T}^{t_i}\delta f_u dt\ ,$$

where the other terms not included within the integration are evaluated at some undetermined time within this time interval. On performing this integration as far as we can yields

$$\delta(\Delta z_j) = \Delta\underline{e}_{jn}^T\delta\underline{r}_u + \underline{e}_{jn}^T\int_{t_i-T}^{t_i}\delta\underline{v}_u dt$$

$$-\left(\Delta\underline{e}_{jn}^T\underline{\underline{R}}_{An} + \underline{e}_{jn}^T\,\Delta\underline{\underline{R}}_{An}\right)\underline{\delta\phi} \tag{22}$$

$$-\underline{e}_{jn}^T\underline{\underline{R}}_{An}\int_{t_i-T}^{t_i}\delta\underline{\omega}_{ib}dt + \int_{t_i-T}^{t_i}\delta f_u dt\ .$$

We arrive at a first order approximation to the temporal difference part if we consider the navigation reference point, the local clock, and attitude changes to be temporally "well behaved" so that $\delta\underline{v}_u$, $\delta\underline{\omega}_{ib}$, and δf_u are regarded to constant over $(t_i-T,\ t_i)$.

Then

$$\delta(\Delta z_j) = \Delta\underline{e}_{jn}^T\,\delta\underline{r}_u + \underline{e}_{jn}^T T\delta\underline{v}_u$$

$$-(\Delta\underline{e}_{jn}^T\underline{\underline{R}}_{An} + \underline{e}_{jn}^T\Delta\underline{\underline{R}}_{An})\ \underline{\delta\phi} \tag{23}$$

$$-\underline{e}_{jn}^T T\underline{\underline{R}}_{An}\,\delta\underline{\omega}_{ib} + T\delta f_u$$

or, more specifically,

$$\begin{aligned}\delta(\Delta z_j(t_i)) &= \Delta \underline{e}^T_{jn}(t_i)\,\delta\underline{r}_u(t_i) + \underline{e}^T_{jn}(t_a)\,T\delta\underline{v}_u \\ &-(\Delta\underline{e}^T_{jn}(t_i)\,\underline{\underline{R}}_{An}(t_b) + \underline{e}^T_{jn}(t_c)\,\Delta\underline{\underline{R}}_{An}(t_i))\,\underline{\delta\phi}(t_i) \\ &-\underline{e}^T_{jn}(t_d)\,TR_{An}(t_e)\,\delta\underline{\omega}_{ib}(t_i) + T\delta f_u(t_i)\,,\end{aligned} \tag{24}$$

where $t_i - T \leq t_a, t_b, t_c, t_d, t_e < t_i$.

Incorporating both pseudo-range and delta-range residuals, the resulting H matrix is

$$\begin{array}{c} \\ \delta z_1 \\ \delta z_2 \\ \vdots \\ \delta z_k \\ \delta(\Delta z_1) \\ \delta(\Delta z_2) \\ \vdots \\ \delta(\Delta z_k) \end{array}
\begin{array}{c}
\begin{array}{cccccc} \delta\underline{r}_u & \delta\underline{v}_u & \underline{\delta\phi} & \delta\underline{\omega}_{ib} & \delta B_u & \delta f_u \end{array} \\
\left[\begin{array}{cccccc}
\underline{e}^T_{1n} & \underline{0}^T & -\underline{e}^T_{1n}\underline{\underline{R}}_{An} & \underline{0}^T & 1 & 0 \\
\underline{e}^T_{2n} & \underline{0}^T & -\underline{e}^T_{2n}\underline{\underline{R}}_{An} & \underline{0}^T & 1 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\underline{e}^T_{kn} & \underline{0}^T & -\underline{e}^T_{Rn}\underline{\underline{R}}_{An} & \underline{0}^T & 1 & 0 \\
\Delta\underline{e}^T_{1n} & \underline{e}^T_{1n}T & -(\Delta\underline{e}^T_{1n}\underline{\underline{R}}_{An} + \underline{e}^T_{1n}\Delta\underline{\underline{R}}_{An}) & -\underline{e}^T_{1n}T\underline{\underline{R}}_{An} & 0 & T \\
\Delta\underline{e}^T_{2n} & \underline{e}^T_{2n}T & -(\Delta\underline{e}^T_{2n}\underline{\underline{R}}_{An} + \underline{e}^T_{2n}\Delta\underline{\underline{R}}_{An}) & -\underline{e}^T_{2n}T\underline{\underline{R}}_{An} & 0 & T \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\Delta\underline{e}^T_{kn} & \underline{e}^T_{kn}T & -(\Delta\underline{e}^T_{kn}\underline{\underline{R}}_{An} + \underline{e}^T_{kn}\Delta\underline{\underline{R}}_{An}) & -\underline{e}^T_{kn}T\underline{\underline{R}}_{An} & 0 & T
\end{array}\right]
\end{array}. \tag{25}$$

If, instead of bringing the H matrix to the antenna phase center, had we derived an H matrix based on the antenna phase center coinciding with the reference point, that is $\underline{\underline{R}}_A=\underline{\underline{0}}$ (but still by using measurement residuals based on antenna-lever-arm corrections), the resulting H matrix would show no coupling between the residuals $\underline{\delta\phi}$ and $\delta\underline{\omega}_{ib}$. By eliminating these noncoupled terms, we have

$$
\begin{array}{c}
\\
\delta z_1 \\ \delta z_2 \\ \vdots \\ \delta z_k \\ \delta(\Delta z_1) \\ \delta(\Delta z_2) \\ \vdots \\ \delta(\Delta z_k)
\end{array}
\begin{array}{c}
\begin{array}{cccc} \delta \underline{r}_u & \delta \underline{v}_u & \delta B_u & \delta f_u \end{array} \\
\begin{bmatrix}
\underline{e}_{1n}^T & \underline{0}^T & 1 & 0 \\
\underline{e}_{2n}^T & \underline{0}^T & 1 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
\underline{e}_{kn}^T & \underline{0}^T & 1 & 0 \\
\Delta \underline{e}_{1n}^T & -\underline{e}_{1n}^T T & 0 & T \\
\Delta \underline{e}_{2n}^T & -\underline{e}_{2n}^T T & 0 & T \\
\vdots & \vdots & \vdots & \vdots \\
\Delta \underline{e}_{kn}^T & -\underline{e}_{kn}^T T & 0 & T
\end{bmatrix}
\end{array},
\tag{26}
$$

which is the usual H matrix encountered when both pseudo-range and delta-range measurements are presented to a Kalman filter implemented in ECEF.

As indicated earlier, when $\Sigma_a = \Sigma_n$ we generally want the terrestrial velocity of the user rather than the velocity as perceived by an observer fixed to Σ_n. In coordinate free form and considering only temporal variation, we define the terrestrial velocity as

$$D_e \overline{r}_u = \overline{v} \cdot \tag{27}$$

By the theorem of Coriolis,

$$
\begin{aligned}
D_e \overline{r}_u &= D_a \overline{r}_u + \overline{\omega}_{ea} \times \overline{r}_u \\
&= D_e \overline{r}_u - \overline{\omega}_{ea} \times \overline{r}_u ,
\end{aligned}
\tag{28}
$$

or, in matrix form

$$
\begin{aligned}
D\underline{r}_u &= \underline{\underline{C}}_e^a (D\underline{r}_u^e) - \underline{\underline{\Omega}}_{ea} \underline{r}_u \\
\underline{v}_u &= v - \underline{\underline{\Omega}}_{ea} \underline{r}_u .
\end{aligned}
\tag{29}
$$

Since we take Σ_a to be the actual frame (Σ_a does not have a nominal counterpart in this analysis), we have

$$\delta \underline{v}_u = \delta \underline{v} - \underline{\underline{\Omega}}_{ea} \delta \underline{r}_u . \tag{30}$$

Using terrestrial velocity $\underline{v}$, (23) becomes

$$\begin{aligned}\delta(\Delta z_j) = {} & (\Delta\underline{e}^T_{jn} - \underline{e}^T_{jn} T \underline{\underline{\Omega}}_{ea})\,\delta\underline{r}_u + \underline{e}^T_{jn} T \delta\underline{v} \\ & - (\Delta\underline{e}^T_{jn} \underline{\underline{R}}_{An} + \underline{e}^T_{jn} \Delta\underline{\underline{R}}_{An})\,\underline{\delta\phi} - \underline{e}^T_{jn} T \underline{\underline{R}}_{An}\,\underline{\delta\omega}_{ib} + T\delta f_u .\end{aligned} \tag{31}$$

To a first order $\Delta\underline{e}^T_{jn} - \underline{e}^T_{jn} T \underline{\underline{\Omega}}_{ea} = (\underline{\underline{C}}^a_e \Delta\underline{e}^e_{jn})^T_,$ which is the temporal change in the satellite unit vector as perceived by an Earth fixed observer and expressed in Σ_a.

We then have as defining the observation matrix H,

$$\begin{array}{cc}
 & \begin{array}{cccccc} \delta\underline{r}_u & \delta\underline{v} & \underline{\delta\phi} & \underline{\delta\omega}_{ib} & \delta B_u & \delta f_u \end{array} \\
\begin{array}{c} \delta z_1 \\ \delta z_2 \\ \vdots \\ \delta z_k \\ \delta(\Delta z_1) \\ \delta(\Delta z_2) \\ \vdots \\ \delta(\Delta z_k) \end{array} &
\begin{bmatrix}
\underline{e}^T_{1n} & \underline{0}^T & -\underline{e}^T_{1n}\underline{\underline{R}}_A & \underline{0}^T & 1 & 0 \\
\underline{e}^T_{2n} & \underline{0}^T & -\underline{e}^T_{2n}\underline{\underline{R}}_A & \underline{0}^T & 1 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\underline{e}^T_{kn} & \underline{0}^T & -\underline{e}^T_{kn}\underline{\underline{R}}_A & \underline{0}^T & 1 & 0 \\
(\underline{\underline{C}}^a_e \Delta\underline{e}^e_{1n})^T & \underline{e}^T_{1n} T & -(\Delta\underline{e}^T_{1n}\underline{\underline{R}}_{An} + \underline{e}^T_{1n}\Delta\underline{\underline{R}}_{An}) & -\underline{e}^T_{1n} T \underline{\underline{R}}_{An} & 1 & T \\
(\underline{\underline{C}}^a_e \Delta\underline{e}^e_{2n})^T & \underline{e}^T_{2n} T & -(\Delta\underline{e}^T_{2n}\underline{\underline{R}}_{An} + \underline{e}^T_{2n}\Delta\underline{\underline{R}}_{An}) & -\underline{e}^T_{2n} T \underline{\underline{R}}_{An} & 0 & T \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
(\underline{\underline{C}}^a_e \Delta\underline{e}^e_{kn})^T & \underline{e}^T_{kn} T & -(\Delta\underline{e}^T_{kn}\underline{\underline{R}}_{An} + \underline{e}^T_{kn}\Delta\underline{\underline{R}}_{An}) & -\underline{e}^T_{kn} T \underline{\underline{R}}_{An} & 0 & T
\end{bmatrix}
\end{array} . \tag{32}$$

If we model $\underline{\delta\omega}_{ib}$ to consist only of gyro bias $\underline{\delta\omega}_B$ and scale factor[4] $\underline{\delta\omega}_K$,

$$\underline{\delta\omega}_{ib} \approx \underline{\underline{C}}^a_{bn}\underline{\delta\omega}^b_B + \underline{\underline{C}}^a_{bn}\underline{\underline{W}}^b\underline{\delta\omega}^b_K , \tag{33}$$

[4] Thus far we have structured the error state elements to be partitioned into 3d vectors and we will continue to do so here for convenience. One may want to model only the x component of scale factor or only certain g-sensitive terms or the like.

where we note the underlining of $\underline{\delta\omega}$ and $\underline{\delta\omega}$ to be single vector quantities and not $\delta(\underline{\cdot})$ and [illegible]

$$
\underline{\underline{W}}^{b} = \begin{bmatrix} \underline{\omega}_{ibx}^{b} & & \\ & \underline{\omega}_{iby}^{b} & \\ & & \underline{\omega}_{ibz}^{b} \end{bmatrix} . \tag{34}
$$

Using terrestrial velocity, we then have for the delta range part

$$
\begin{aligned}
\delta(\Delta z) = {} & (\Delta\underline{e}^{T} - \underline{e}^{T}\, T\underline{\underline{\Omega}})\,\delta\underline{r} + \underline{e}^{T}\, T\delta\underline{v} \\
& - (\Delta\underline{e}^{T}\underline{\underline{R}}_{A} + \underline{e}^{T}\Delta\underline{\underline{R}}_{A})\,\underline{\delta\phi} \\
& - \underline{e}^{T}\, T\underline{\underline{R}}_{A}\underline{\underline{C}}\,\underline{\delta\omega} - \underline{e}^{T}\, T\underline{\underline{R}}_{A}\underline{\underline{C}}\,\underline{\underline{W}}\,\underline{\delta\omega} + T\delta f \, .
\end{aligned} \tag{35}
$$

The H matrix of expanded column dimensionality would then be constructed with these additional terms.

2.4 Higher Order Terms

Up to this point, we have used only a first order approximation to the temporal difference process $\Delta(\cdot)$. A first order approximation is, of course, used for the spatial difference process $\delta(\cdot)$, to obtain a linear form for the Kalman filter. While we retain the first order approximation to the spatial difference process, we here briefly investigate whether other coupling terms may arise from a higher order approximation to the temporal difference part.

Repeating (22)[5],

$$\delta(\Delta z_j) = \Delta\underline{e}_{jn}^T \delta\underline{r}_u + \underline{e}_{jn}^T \int_{t_i-T}^{t_i} \delta\underline{v}_u dt$$

$$-\left(\Delta\underline{e}_{jn}^T \underline{\underline{R}}_{An} + \underline{e}_{jn}^T \Delta\underline{\underline{R}}_{An}\right)\underline{\delta\phi} \tag{22}$$

$$-\underline{e}_{jn}^T \underline{\underline{R}}_{An} \int_{t_i-T}^{t_i} \delta\underline{\omega}_{ib} dt + \int_{t_i-T}^{t_i} \delta f_u dt \, .$$

In taking $\delta\underline{v}_u$, $\delta\underline{\omega}_{ib}$, and δf_u to be constant over $(t_i-T,\ t_i)$ to arrive at (23) is tantamount to approximating each of these functions of time by the first term of a Taylor series expansion about t_i. We now look at additional terms.

Considering only temporal variation, a Taylor series expansion of $\delta\underline{V}_u$ about t_i and by interchanging δ and time differentiation yields

$$\delta\underline{v}_u(t) = \delta\underline{v}_u(t_i) + \delta\left(\frac{\partial\underline{v}_u}{\partial t}\Big|_{t=t_i}\right)(t-t_i) + \text{h.o.t}, \tag{36}$$

Relating to coordinate free form,

$$D\underline{v}_u \longleftrightarrow D_a^2\overline{r}_u = D_i^2\overline{r}_u - \overline{\omega}_{ia}\times(\overline{\omega}_{ia}\times\overline{r}_u) - 2\overline{\omega}_{ia}\times D_a\overline{r}_u$$

$$D_a\overline{r}_u = D_e\overline{r}_u - \overline{\omega}_{ae}\times\overline{r}_u = \overline{v}_u \tag{37}$$

$$\overline{a} \triangleq D_i^2\overline{r}_u \quad , \quad \overline{v} = D_e\overline{r}_u \, ,$$

$\overline{a}$ is the inertial acceleration and $\overline{v}$, as previously noted, is the terrestrial velocity.

---

[5] It will be more convenient to initially use $\underline{v}_u$ in this development rather than terrestrial velocity $\underline{v}$ and then make the adjustments. We seek to arrive at an acceleration error state that may be related to existing accelerometer error state elements of the state vector.

We then have

$$\delta(D\underline{v}_u) = \delta\left(\underline{a} - \underline{\underline{\Omega}}_{ia}\underline{\underline{\Omega}}_{ia}\underline{r}_u - 2\underline{\underline{\Omega}}_{ia}(\underline{v} - \underline{\underline{\Omega}}_{ea}\underline{r}_u)\right)$$
$$= \delta\underline{a} - 2\underline{\underline{\Omega}}_{ia}\delta\underline{v} - (\underline{\underline{\Omega}}^2_{ia} - 2\underline{\underline{\Omega}}_{ia}\underline{\underline{\Omega}}_{ea})\delta\underline{r}_u \,. \tag{38}$$

By using (30), (36), and (38) we have

$$\delta\underline{v}_u(t) = \delta\underline{v}(t_i) - \underline{\underline{\Omega}}_{ea}\delta\underline{r}_u(t_i)$$
$$+ (t-t_i)\,[\delta\underline{a}(t_i) - 2\underline{\underline{\Omega}}_{ia}\delta\underline{v}(t_i) - (\underline{\underline{\Omega}}^2_{ia} - 2\underline{\underline{\Omega}}_{ia}\underline{\underline{\Omega}}_{ea})\delta\underline{r}_u(t_i)]\,, \tag{39}$$

and

$$\delta\underline{v}_u(t) = -\left[\underline{\underline{\Omega}}_{ea} + (t-t_i)(\underline{\underline{\Omega}}^2_{ia} - 2\underline{\underline{\Omega}}_{ia}\underline{\underline{\Omega}}_{ea})\right]\delta\underline{r}_u(t_i)$$
$$+ \left[\underline{\underline{I}} - 2(t-t_i)\underline{\underline{\Omega}}_{ia}\right]\delta\underline{v}(t_i)$$
$$+ (t-t_i)\,\delta\underline{a}(t_i)\,. \tag{40}$$

We then have the second order approximation for the integral

$$\int_{t_i-T}^{t_i}\delta\underline{v}_u dt = -\left[\underline{\underline{\Omega}}_{ea}T - (\underline{\underline{\Omega}}^2_{ia} - 2\underline{\underline{\Omega}}_{ia}\underline{\underline{\Omega}}_{ea})\frac{T^2}{2}\right]\delta\underline{r}_u(t_i)$$
$$+ (\underline{\underline{I}}T + \underline{\underline{\Omega}}_{ia}T^2)\,\delta\underline{v}(t_i) - \frac{T^2}{2}\delta\underline{a}(t_i)\,. \tag{41}$$

In coordinate free form,

$$\overline{a} = D_i^2\overline{r}_u = \overline{f} + \overline{G}$$

or, in matrix form, the corresponding errors are

$$\delta\underline{a} = \delta\underline{f} + \delta\underline{G}\,,$$

where $\underline{f}^b$ is the specific force vector which is measured by the accelerometers and $\underline{G}$ is the gravitation vector. As $\delta\underline{G}$ is not modeled here, we delete this term. If we take $\delta\underline{\omega}_{ib}$ and δf_u to be constant over $(t_i-T,\ t_i)$, from (22) and (41) we have

$$
\begin{aligned}
\delta(\Delta z)_j) = & \left\{ \Delta \underline{e}^T_{jn} - \underline{e}^T_{jn} \left[\underline{\underline{\Omega}}_{ia} T - (\underline{\underline{\Omega}}^2_{ia} - 2 \underline{\underline{\Omega}}_{ia} \underline{\underline{\Omega}}_{ae}) \frac{T^2}{2} \right] \right\} \delta \underline{r}_u \\
& + \underline{e}^T_{jn} (\underline{\underline{I}} T + \underline{\underline{\Omega}}_{ia} T^2) \delta \underline{v} \\
& - (\Delta \underline{e}^T_{jn} \underline{\underline{R}}_{An} + \underline{e}^T_{jn} \Delta \underline{\underline{R}}_{An}) \underline{\delta \phi} \\
& - \underline{e}^T_{jn} \underline{\underline{R}}_{An} T \delta \underline{\omega}_{ib} \\
& - \underline{e}^T_{jn} \frac{T^2}{2} \delta \underline{f} \\
& + T \delta f_u .
\end{aligned} \qquad (42)
$$

Though not pursued further here, $\delta \underline{f}^b$ is expressible in terms of accelerometer error states such as bias, scale factor, misalignments, etc.

A similar approach could be taken with regard to $\delta \underline{\omega}_{ib}$, not making the assumption that it is constant over $(t_i - T, t_i)$. However, an angular acceleration error state would result that is not modeled. Moreover, if it were modeled, an issue arises on the time propagation of the corresponding total state because we do not generally incorporate corresponding sensors that would aid its propagation in a high dynamic environment.

2.5 Comparison To Standard Attitude Measurement Techniques

Although a comparison to an exhaustive list of other GPS-based attitude determination methods is beyond the scope of this report, a quick comparison of the method described in this report to "interferometric" attitude determination methods (Van Graas & Braasch, 1991-1992) may be useful at this point. We can construct a configuration which affords a convenient general comparison between the technique presented here and the more familiar attitude measurement techniques. For purposes of comparison we take any two noncoincident GPS antennas of the "standard" interferometric method and the set of their phase differences for each of four GPS satellites to comprise a *spatial* GPS Interferometer. A set of delta-range measurements taken over the same time interval for four GPS satellites using a single antenna in effect provides a *temporal* version of a GPS Interferometer as defined above. While more restrictive than need be due to use of recursive estimation techniques that preserve information by propagating it forward, the typical implementation of a conventional five channel GPS receiver having pseudo-range and delta-range measurements would, in fact, conform to such a configuration.

A brief comparison between the technique described in this report (denoted as "Delta-Range-Based method") and standard GPS interferometric methods is provided in table 1.

Table 1. Comparison of standard interferometric GPS-based attitude measurement techniques to delta-range-based method.

Standard (Multiple Antennas)	DR-Based (Single Antenna)
Requires multiple antennas and a special GPS receiver design. No requirement on host-vehicle motion relative to GPS satellite constellation.	Requires only a single antenna and a standard GPS receiver having pseudo-range and delta-range measurement capability. Requires changing attitude of host vehicle relative to GPS satellite constellation. Integrated GPS/INS generally needed for good performance.
Any two noncoincident antennas receiving carriers of four GPS satellites and the single difference of their phases constitute a *spatial* GPS Interferometer.	Each set of DR measurements for four GPS satellites using a single antenna on host vehicle undergoing attitude changes may be viewed as constructing *temporally* a GPS Interferometer having as antennas the start and stop antenna locations. The accumulated phases of the DR measurement set are equivalent to the set of differences of received phases at the start and stop antenna locations.
A minimum of two nonparallel GPS Interferometers (a minimum of three noncolinear antennas) is required for attitude determination at any time.	A minimum of two nonparallel equivalent GPS Interferometers (two DR measurement sets) required for attitude determination with INS propagation of information over the DR time period. Attitude is determined for end of period and is propagated forward by INS.
The satellite phase errors are removed by the single *spatial* difference in phase between the two antennas of the GPS Interferometer.	Removal of satellite phase errors is inherent to each DR measurement.

Standard (Multiple Antennas)	DR-Based (Single Antenna)
Ambiguity resolution for integer cycles must be accomplished for the single phase differences. Use of redundancy, search procedures, triple differences, etc., are required.	No ambiguities in each DR measurement provided no loss of carrier lock or cycle slip in GPS receiver channel associated with GPS satellite.
May require calibration and compensation for antenna induced error due to phase pattern differences between the two antennas constituting the GPS Interferometer.	Fundamental phase change due to attitude changes in single antenna must be compensated for. Antenna must also be calibrated for phase pattern anomalies.
Double difference (difference between two independent single differences) required to remove receiver timing errors, electrical path bias errors, etc.	Use of single antenna and standard DR measurement obviate the need for double differences.
Direct solution of attitude performed.	Recursive estimation (using Kalman Filter) of attitude errors relative to nominal trajectory established by INS and its corrections (resets) is performed.

3.0 STATE-SPACE FORMULATION

3.1 Introduction

In developing a measurement model, the method presented in section 2 has taken a direct approach. The strength of such an approach is that it has remained independent of any other analysis and, thereby, has been exploratory in seeking coupling between pseudo-range and delta-range measurement residuals and the error states. The results for the pseudo-range residuals need no further elaboration and are useful as derived. With regard to delta-range residuals, however, a weakness is that it has not been established where certain terms in (22) need to be evaluated to be equivalent to (21). For example, when the time interval T over which the temporal difference is taken is small such that the change in $\underline{\underline{R}}_A$ is small, the problem is minimal and any reasonable choice of evaluation time will provide useful results. For greater efficiency in coupling to the attitude errors, however, we wish to maximize $\Delta\underline{\underline{R}}_A$. The choice of where to best evaluate $\underline{\underline{R}}_A$ in the interval (t_i-T, t_i) is uncertain.

In this section, we utilize a presumed existing dynamic model to develop the temporal differences required for the delta-range relations through backward transition matrices. The strength of this approach is that there is no uncertainty as to where to evaluate the terms in the resulting relations. The weakness, however, is that it depends on a previously developed dynamics model and is subject to any inadequacies in this model. In particular, uncertainty in the backward time propagation contributes to uncertainty in the measurement model.

3.2 Reference Frames

The previous approach required careful identification and manipulation of representation in various reference frames. The state-space formulation requires only minimal consideration. The only assumption made with regard to reference frames for the error state vector corresponding to the previously developed dynamics matrix is that position errors and attitude errors are represented in the same selected preferred reference frame. In this development, position vectors and the skew-symmetric matrix corresponding to the antenna-lever-arm position vector will also be represented in this preferred reference frame. Otherwise, the error state vector can mix and match reference frames, the only requirement being consistency with the previously developed dynamics model.

3.3 State Equations

The system model will be taken to be nonlinear but with additive noise (Maybeck, 1979; Anderson & Moore, 1979). In particular, the total state dynamic model is taken to be continuous and given by[6]

$$\dot{\underline{x}}(t) = \underline{f}[\underline{x}(t), t] + \underline{\underline{G}}(t)\underline{w}(t), \tag{43}$$

where $\underline{w}(t)$ is a zero-mean white Gaussian noise process. For the purpose of either a linearized or extended Kalman filter, we assume a nominal trajectory of similar form

$$\dot{\underline{x}}_n(t) = \underline{f}[\underline{x}_n(t), t]. \tag{44}$$

The spatial difference process

$$\delta\underline{x}(t) = \underline{x}(t) - \underline{x}_n(t) \tag{45}$$

has the first order approximation

$$\delta\dot{\underline{x}}(t) = \underline{\underline{F}}[t; \underline{x}_n(t)]\,\delta\underline{x}(t) + \underline{\underline{G}}(t)\underline{w}(t), \tag{46a}$$

where

$$\underline{\underline{F}}[t; \underline{x}_n(t)] \triangleq \frac{\partial \underline{f}(\underline{x}, t)}{\partial \underline{x}}\Big|_{\underline{x}=\underline{x}_n(t)}. \tag{46b}$$

In the case of an extended Kalman filter coupling a GPS receiver with an IMU, the nominal trajectory is updated (reset) following measurement update to the new total state estimate so that the error state, now reset to zero, need not be propagated. As indicated in the previous development, propagation of the total state is largely accomplished through use of IMU data.

The discrete time measurements are taken to be a nonlinear function of the state with an additive white noise sequence.

$$\underline{z}(t_i) = \underline{h}[\underline{x}(t_i), t_i] + \underline{v}(t_i). \tag{47}$$

Associated with the nominal trajectory is the sequence of nominal measurements

$$\underline{z}_n(t_i) = \underline{h}[\underline{x}_n(t_i), t_i]. \tag{48}$$

---

[6] It is noted that (43) implies definition of "attitude total states." Attitude total states are not usually defined. The formalism in (43), however, conveniently leads to the definition of attitude error states.

The spatial difference process

$$\delta \underline{z}(t_i) = \underline{z}(t_i) - \underline{z}_n(t_i) = \underline{h}[\underline{x}(t_i), t] - \underline{h}[\underline{x}_n(t_i), t;] + v(t_i) \tag{49}$$

has the first order approximation

$$\delta \underline{z}(t_i) = \underline{\underline{H}}[t_i; \underline{x}_n(t_i)]\,\delta \underline{x}(t_i) + \underline{v}(t_i), \tag{50}$$

where

$$\underline{\underline{H}}[t_i; \underline{x}_n(t_i)] \triangleq \frac{\partial \underline{h}[\underline{x}(t_i), t_i]}{\partial \underline{x}}\Big|_{\underline{x}=\underline{x}_n(t_i)}.$$

If we express

$$\underline{h}'[\delta \underline{x}(t_i), \underline{x}(t_i) t_i] = \underline{h}[\underline{x}(t_i) t_i] - \underline{h}[\underline{x}_n(t_i), t_i], \tag{51}$$

where we treat $\delta \underline{x}$ as an independent variable, then

$$\underline{\underline{H}}[t_i; \underline{x}_n(t_i] = \frac{\partial \underline{h}'[\delta \underline{x}, \underline{x}, t_i]}{\partial \delta \underline{x}}\Big|_{\delta \underline{x}=0}. \tag{52}$$

Further discussion and proof is given in Appendix A.

For this problem we can express

$$\underline{h}(\underline{x}, t_i) = \begin{bmatrix} h_1(\underline{x}, t_i) \\ h_2(\underline{x}, t_i) \\ \vdots \\ h_m(\underline{x}, t_i) \end{bmatrix} \text{and} \quad \underline{h}'(\delta \underline{x}, \underline{x}, t_i) = \begin{bmatrix} h_1'(\delta \underline{x}, \underline{x}, t_i) \\ h_2'(\delta \underline{x}, \underline{x}, t_i) \\ \vdots \\ h_m'(\delta \underline{x}, \underline{x}, T_i) \end{bmatrix} \tag{53}$$

so that the jth row of $\underline{\underline{H}}[t_i; \underline{x}_n(t_i)]$ is given by

$$h_j^T[t_i; \underline{x}_n(t_i)] = \frac{\partial h_j(\underline{x}, t_i)}{\partial \underline{x}}\Big|_{\underline{x}=\underline{x}_n(t)} = \frac{\partial h_j'(\delta \underline{x}, \underline{x}, t_i)}{\partial(\delta \underline{x})}\Big|_{\delta \underline{x}=0}. \tag{54}$$

3.4 Approach

The approach is to develop a spatial difference of the temporal difference process over the time interval $(t_i - T,\ t_i)$ for the jth delta range (i.e., the delta range for the jth satellite). The delta-range residuals are determined in terms of the geometry

in figure 1. We then use a previously developed linearized dynamics model to express error state terms having time arguments of t_i-T by the product of the backward transition matrix (appropriately premultiplied by other matrices to isolate the required terms) and the error state vector having the time argument t_i. We then have a "row" of the measurement model utilizing the argument $\delta\underline{x}$. We then apply the results of Appendix A and (54) to obtain the corresponding row of the H matrix in the perturbation model of (50).

3.5 Delta Range

From figure 1 we have the vector from the jth satellite to the antenna phase center at any time t and represented in some preferred frame Σ_a as

$$\underline{r}_j(t) = \underline{r}_u(t) + \underline{r}_A(t) - \underline{r}_{sj}(t) . \tag{55}$$

The rf range $R_j(t)$ from the antenna phase center to the jth satellite (or that from the jth satellite to the phase center as we take range to be a positive value) must account for the range equivalent user clock offset

$$R_j(t) = \|\underline{r}_j(t)\| + B_u(t) . \tag{56}$$

We also have nominal rf range corresponding to the nominal trajectory

$$R_{jn}(t) = \|\underline{r}_{jn}(t)\| + B_{un}(t) . \tag{57}$$

For the rf range and nominal rf range at time t-T, we would have the same forms as (56) and (57) but with arguments t-T.

The measured delta range as a discrete time process (at t_i) and corrupted by measurement noise is[7]

$$z_{DRj}(t_i) = R_j(t_i) - R_j(t_i-T) + v_{DRj}(t_i) . \tag{58}$$

[7] Although (58) has the appearance of the difference of two ranges, it must be remembered that the measurement is not obtained as the difference of two pseudo-range measurements but rather as a receiver measure of the accumulated carrier phase (integrated Doppler shift) from t_i-T to t_i. In this regard, $v_{DRj}(t_i)$ is the discrete measurement noise process involving the effective carrier tracking loop noise over the same interval.

The nominal delta range corresponding to the nominal trajectory is

$$z_{DRjn}(t_i) = R_{jn}(t_i) - R_{jn}(t_i - T). \tag{59}$$

The delta-range measurement residual discrete process is then

$$\begin{aligned}\delta z_{DRj}(t_i) &= z_{DRj}(t_i) - z_{DRjn}(t_i) \\ &= R_j(t_i) - R_{jn}(t_i) - [R_j(t_i - T) - R_{jn}(t_i - T)] + v_{DRj}(t_i).\end{aligned} \tag{60}$$

As for any time t

$$R_{jn}(t) = \| \underline{r}_{un}(t) + \underline{r}_{An}(t) - \underline{r}_{sj}(t) \| + B_{un}(t) \tag{61}$$

and as spatial difference processes using $\delta(\cdot) = (\cdot) - (\cdot)_n$, are

$$\begin{aligned}\delta \underline{r}_u(t) &= \underline{r}_u(t) - \underline{r}_{un}(t) \\ \delta \underline{r}_A(t) &= \underline{r}_A(t) - \underline{r}_{An}(t) \\ \delta B_u(t) &= B_u(t) - B_{un}(t),\end{aligned} \tag{62}$$

we have

$$\begin{aligned}R_{jn}(t) &= \| \underline{r}_u(t) - \delta \underline{r}_u(t) + \underline{r}_A(t) - \delta \underline{r}_A(t) - \underline{r}_{sj}(t) \| + B_u(t) - \delta B_u(t) \\ &= \| \underline{r}_j(t) - \delta \underline{r}_u(t) - \delta \underline{r}_A(t) \| + B_u(t) - \delta B_u(t).\end{aligned} \tag{63}$$

Hence the delta range measurement residual discrete process for the jth satellite is

$$\begin{aligned}&\delta z_{DRj}(t_i) = \| \underline{r}_j(t_i) \| - \| \underline{r}_j(t_i) - \delta \underline{r}_u(t_i) - \delta \underline{r}_A(t_i) \| + \delta B_u(t_i) \\ &- \left\{ \| \underline{r}_j(t_i - T) \| - \| \underline{r}_j(t_i - T) - \delta \underline{r}_u(t_i - T) - \delta \underline{r}_A(t_i - T) \| + \delta B_u(t_i - T) \right\} \\ &+ v_{DRj}(t_i).\end{aligned} \tag{64}$$

Taking the preferred reference frame Σ_a to be a perfectly known frame and the host vehicle body frame Σ_r to be uncertain (Σ_a

has no nominal counterpart but Σ_b does, designated here as Σ_{bn}) we have

$$\underline{r}_A(t) = \underline{\underline{C}}_b^a(t)\,\underline{r}_A^b\,. \tag{65}$$

Even though Σ_b is uncertain, $\underline{r}_A^b$, the location of the antenna phase center in the host vehicle frame, is taken to be known perfectly.

We then have

$$\delta\underline{r}_A(t) = \underline{r}_A(t) - \underline{r}_{An}(t) = \underline{\underline{C}}_b^a(t)\,\underline{r}_A^b - \underline{\underline{C}}_{bn}^a(t)\,\underline{r}_{An}^{bn}\,. \tag{66}$$

We also note that $\underline{r}_{An}^{bn}$ corresponding to the nominal trajectory is also perfectly known and, in fact, is equal to $\underline{r}_A^b$. The antenna phase center for the nominal host frame has the same relative location as the actual antenna phase center for the actual host frame. Then in the process of defining the skew-symmetric matrix of attitude error (the difference between Σ_b and Σ_{bn}),

$$\begin{aligned}
\delta\underline{r}_A &= \left[\underline{\underline{C}}_b^a(t) - \underline{\underline{C}}_{bn}^a(t)\right]\underline{r}_{An}^{bn} = \underline{\underline{C}}_{bn}^a(t)\left[\underline{\underline{C}}_b^{bn}(t) - \underline{\underline{I}}\right]\underline{r}_{An}^{bn} \\
&\underline{\underline{\delta\Psi}}^{bn}(t) \triangleq \underline{\underline{\delta\Psi}}_{bn,b}^{bn}(t) \triangleq \underline{\underline{C}}_b^{bn}(t) - \underline{\underline{I}} \\
\delta\underline{r}_A(t) &= \underline{\underline{C}}_{bn}^a(t)\,\underline{\underline{\delta\Psi}}^{bn}(t)\,\underline{\underline{C}}_a^{bn}(t)\,\underline{\underline{C}}_{bn}^a(t)\,\underline{r}_{An}^{bn}(t) \\
&= \underline{\underline{\delta\Psi}}^a(t)\,\underline{r}_{An}^a(t) = \underline{\underline{\delta\Psi}}(t)\,\underline{r}_{An}(t)\,.
\end{aligned} \tag{67}$$

Using the skew-symmetric form $\underline{\underline{R}}_{An}(t)$ for vector $\underline{r}_{An}(t)$, we have from (11)

$$\begin{aligned}
\delta\underline{r}_A(t) &= -\underline{\underline{R}}_{An}(t)\,\underline{\delta\phi}(t)\,, \\
\underline{\underline{R}}_{An} &= \begin{bmatrix} 0 & -r_{Azn}(t) & r_{Ayn}(t) \\ r_{Azn}(t) & 0 & -r_{Axn}(t) \\ -r_{Ayn}(t) & r_{Axn}(t) & 0 \end{bmatrix}.
\end{aligned} \tag{68}$$

We then have for the discrete time measurement residual process

$$\delta z_{DRj}(t_i) = \|\underline{r}_j(t_i)\| - \|\underline{r}_j(t_i) - \delta\underline{r}_u(t_i) + \underline{\underline{R}}_{An}(t_i)\underline{\delta\phi}(t_i)\| + \delta B_u(t_i)$$

$$- \left\{ \|\underline{r}_j(t_i - T)\| - \|\underline{r}_j(t_i - T) - \delta\underline{r}_u(t_i - T) + \underline{\underline{R}}_{An}(t_i - T)\underline{\delta\phi}(t_i - T)\| + \delta B_u(t_i - T) \right\}$$

$$+ v_{DRj}(t_i) . \tag{69}$$

We define 3 x n matrices $\underline{\underline{K}}_p$ and $\underline{\underline{K}}_{\phi}$ and row vector (1 x n matrix) $\underline{k}_B^T$ such that we can relate to the full error state vector at measurement time t_i.

$$\begin{aligned} \delta\underline{r}_u(t_i) &= \underline{\underline{K}}_p \delta\underline{x}(t_i) \\ \underline{\delta\phi}(t_i) &= \underline{\underline{K}}_{\phi} \delta\underline{x}(t_i) \\ \delta B_u(t_i) &= \underline{k}_B^T \delta\underline{x}(t_i) \end{aligned} \tag{70}$$

At measurement time t_i-T, we obtain the corresponding quantities but using the backward transition matrix

$$\begin{aligned} \delta\underline{r}_u(t_i - T) &= \underline{\underline{K}}_p \left[\Phi(t_i - T, t_i)\, \delta\underline{x}(t_i) + \underline{w}_D(t_i - T) \right] \\ \underline{\delta\phi}(t_i - T) &= \underline{\underline{K}}_{\phi} \left[\Phi(t_i - T, t_i)\, \delta\underline{x}(t_i) + \underline{w}_D(t_i - T) \right] \\ \delta B_u(t_i - T) &= \underline{k}_B^T \left[\Phi(t_i - T, t_i)\, \delta\underline{x}(t_i) + \underline{w}_D(t_i - T) \right] , \end{aligned} \tag{71}$$

where $\underline{w}_D(t_i - T)$ is the driven response at t_i-T due to the presence of the white noise in (43) during the backward interval from t_i to t_i-T. $\underline{w}_D(t_i - T)$, by virtue of the white noise in the continuous model, is a white noise sequence. It is here that uncertainty in the dynamics model is introduced in the measurement model.

The discrete delta-range measurement residual process is now

$$\delta z_{DRj}(t_i) = \| \underline{r}_j(t_i) \| - \| \underline{r}_j(t_i) - \left[\underline{\underline{K}}_p - \underline{\underline{R}}_{An}(t_i)\underline{\underline{K}}_\phi\right]\delta\underline{x}(t_i) \|$$

$$-\left\{ \| \underline{r}_j(t_i-T) \| - \| \underline{r}_j(t_i-T) - \left[\underline{\underline{K}}_p - \underline{\underline{R}}_{An}(t_i-T)\underline{\underline{K}}_\phi\right] \left[\underline{\underline{\Phi}}(t_i-T), t_i) \delta\underline{x}(t_i) + \underline{w}_D(t_i-T)\right] \| \right\}$$

$$+\underline{k}_B^T \left\{ \left[\underline{\underline{I}} - 1\Phi(t_i-T, t_i)\right]\delta\underline{x}(t_i) + \underline{w}_D(t_i-T) \right\}$$

$$+v_{DRj}(t_i). \tag{72}$$

Except for the noise introduced by uncertainty in the dynamics model in the form of $\underline{w}_D(t_i-T)$, this is in the form given in Appendix A

$$\delta z_{DRj}(t_i) = h_j'[\delta\underline{x}(t_i), \underline{x}(t_i), t_i] + v_{DRj}. \tag{73}$$

This form is obtained by boosting $v_{DR}(t_i)$ by an appropriate amount and removing $\underline{w}_D(t_i-T)$. In reality, this is accomplished by an increase of the corresponding term in the measurement noise covariance matrix.[8]

From Appendix A we have

$$\delta \underline{z}_{DRj}(t_i) = \frac{\partial h_j'(\delta\underline{x}, \underline{x}, t_i)}{\partial(\delta\underline{x})}\Big|_{\delta\underline{x}=0} \delta\underline{x}(t_i) + v_{DRj}(t_i) + \text{h.o.t.}, \tag{74}$$

where $\frac{\partial h_j'(\delta\underline{x}, \underline{x}, t_i)}{\partial(\delta\underline{x})} = \underline{h}_j^{T}{}''[t_i, \underline{x}_n(t_i)]$,is the jth row of the observation matrix.

---

[8] It is to be noted that $w_D(t_i-T)$ has the argument t_i-T. The measurement noise covariance matrix, which needs to be modified, however, has the argument t_i. This added noise contribution does not correspond to any physical noise process in the GPS receiver but reflects additional uncertainty in the adequacy of the resulting measurement model introduced by using the assumed dynamics model in the development of the measurement model.

Noting that $\|\underline{r}_j(t_i)\|$ and $\|\underline{r}_j(t_i-T)\|$ are not functions of $\delta\underline{x}$ and taking $\underline{w}_D(t_i-T)=\underline{0}$ (by accounting for this additional uncertainty in measurement noise covariance matrix),

$$
\begin{aligned}
\underline{h}_j^{T\prime\prime}[t_i;\underline{x}_n(t_i] = &-\frac{\partial\|\underline{r}_j(t_i)-[\underline{\underline{K}}_p-\underline{\underline{R}}_{An}(t_i)\underline{\underline{K}}_\phi]\,\delta\underline{x}\|}{\partial(\delta\underline{x})}\Big|_{\delta\underline{x}=\underline{0}} \\
&+\frac{\partial\|\underline{r}_j(t_i-T)-\underline{\underline{K}}_p-\underline{\underline{R}}_{An}(t_i-T)\underline{\underline{K}}_\phi]\underline{\underline{\Phi}}(t_i-T,t_i)\,\delta\underline{x}\|}{\partial(\delta\underline{x})}\Big|_{\delta\underline{x}=\underline{0}} \\
&+\frac{\partial\underline{k}_B^T[\underline{\underline{I}}_{nxn}-\underline{\underline{\Phi}}(t_i-T,t_i)]\,\delta\underline{x}}{\partial(\delta\underline{x})}\Big|_{\delta\underline{x}=\underline{0}}\,.
\end{aligned} \tag{75}
$$

By using the results of Appendix B, and since for $\delta\underline{x}=0, \underline{x}=\underline{x}_n$ and $\underline{r}_j=\underline{r}_{jn}$

$$
\begin{aligned}
\underline{h}_j^{T\prime\prime}[t_i;\underline{x}_n(t_i)] = &\frac{\underline{r}_{jn}^T(t_i)[\underline{\underline{K}}_p-\underline{\underline{R}}_{An}(t_i)\underline{\underline{K}}_\phi]}{\|\underline{r}_{jn}(t_i)\|} \\
&-\frac{\underline{r}_{jn}^T(t_i-T)[\underline{\underline{K}}_p-\underline{\underline{R}}_{An}(t_i-T)\underline{\underline{K}}_\phi]\underline{\underline{\Phi}}(t_i-T,t_i)}{\|\underline{r}_{jn}(t_i-T)\|} \\
&+\underline{k}_B^T[\underline{\underline{I}}_{nxn}-\underline{\underline{\Phi}}(t_i-T,t_i)]
\end{aligned} \tag{76}
$$

$$
\text{As,}\quad \frac{\underline{r}_{jn}^T(t)}{\|\underline{r}_{jn}^T(t)\|} = \underline{e}_{jn}^T(t)
$$

$$
\begin{aligned}
\underline{h}_j^{T\prime\prime}[t_i;\underline{x}_n(t_i)] = &\,\underline{e}_{jn}^T(t_i)[\underline{\underline{K}}_p-\underline{\underline{R}}_{An}(t_i)\underline{\underline{K}}_\phi] \\
&-\underline{e}_{jn}^T(t_i-T)[\underline{\underline{K}}_p-\underline{\underline{R}}_{An}(t_i-T)\underline{\underline{K}}_\phi]\underline{\underline{\Phi}}(t_i-T,t_i) \\
&+\underline{k}_B^T[\underline{\underline{I}}_{nxn}-\underline{\underline{\Phi}}(t_i-T,t_i)]\,.
\end{aligned} \tag{77}
$$

If we approximate the backward transition matrix by the first order form

$$
\underline{\underline{\Phi}}(t_i-T,t_i) = \underline{\underline{I}}_{nxn}-\underline{\underline{F}}[t_i;\underline{x}_n(t_i)]T, \tag{78}
$$

then

$$\begin{aligned}\underline{h}_j^{T''}[t_i;\underline{x}_n(t_i)] &= \underline{e}_{jn}^T(t_i)[\underline{\underline{K}}_p - \underline{\underline{R}}_{An}(t_i)\underline{\underline{K}}_\phi] \\ &\quad -\underline{e}_{jn}^T(t_i - T)[\underline{\underline{K}}_p - \underline{\underline{R}}_{An}(t_i - T)\underline{\underline{K}}_\phi][\underline{\underline{I}} - \underline{\underline{F}}[t_i;\underline{x}_n(t_i)]]T \\ &\quad +\underline{k}_B^T\underline{\underline{F}}[t_i;\underline{x}_n(t_i)]T.\end{aligned} \tag{79}$$

The 3xn matrices $\underline{\underline{K}}_p$ and $\underline{\underline{K}}_\phi$ and the row vector $\underline{k}_B^T$ that extract the position error, attitude error and user clock error from the full error state vector necessarily depend on how the full error state vector is constructed but are readily derived.

Rearranging (79), we have for the row corresponding to the delta-range measurement residual for the jth satellite

$$\begin{aligned}\underline{h}_j^{T''}[t_i;\underline{x}_n(t_i)] &= [\underline{e}_{jn}^T(t_i) - \underline{e}_{jn}^T(t_i - T)]\underline{\underline{K}}_p \\ &\quad -[\underline{e}_{jn}^T(t_i)\underline{\underline{R}}_{An}(t_i) - \underline{e}_{jn}^T(t_i - T)\underline{\underline{R}}_{An}(t_i - T)]\underline{\underline{K}}_\phi \\ &\quad +\underline{e}_{jn}^T(t_i - T)\underline{\underline{K}}_p\underline{\underline{F}}[t_i;\underline{x}_n(t_i)]T \\ &\quad -\underline{e}_{jn}^T(t_i - T)\underline{\underline{R}}_{An}(t_i - T)\underline{\underline{K}}_\phi\underline{\underline{F}}[t_i;\underline{x}_n(t_i)]T \\ &\quad +\underline{k}_B^T\underline{\underline{F}}[t_i;\underline{x}_n(t_i)]T.\end{aligned} \tag{80}$$

The best way to compare this method to the first method is to see how the error state vector is coupled to a delta-range measurement residual, by using Σ_a = ECEF as an example, we briefly examine the effect of each of the five terms of (80) for the product $\underline{h}''^T_j[t_i;\underline{x}_n(t_i)]\delta\underline{x}(t_i)$. We then compare with that which would be obtain using a row of (25) that corresponds to a delta range measurement residual. To obtain some of the terms in (80), of course, we must have a dynamics matrix.

For the first term of (80) we have

$$\begin{aligned}[\underline{e}_{jn}^T(t_i) - \underline{e}_{jn}^T(t_i - T)]\underline{\underline{K}}_p\delta\underline{x}(t_i) &= [\underline{e}_{jn}^T(t_i) - \underline{e}_{jn}^T(t_i - T)]\delta\underline{r}_u(t_i) \\ &= \Delta\underline{e}_{jn}^T(t_i)\delta\underline{r}_u(t_i).\end{aligned} \tag{81}$$

The result is identical to that for the corresponding term in (25). In this case there is no problem as to which time to evaluate any part.

For the second term of (80) the result is the same as for the corresponding term in (25) except here the time of evaluation is quite clear.

$$
\begin{aligned}
-&\left[\underline{e}_{jn}^{T}(t_i)\underline{\underline{R}}_{An}(t_i)-\underline{e}_{jn}^{T}(t_i-T)\underline{\underline{R}}_{An}(t_i-T)\right]\underline{\underline{K}}_{\phi}\delta\underline{x}(t_i)\\
&= -\left[\underline{e}_{jn}^{T}(t_i)\underline{\underline{R}}_{An}(t_i)-\underline{e}_{jn}^{T}(t_i-T)\underline{\underline{R}}_{An}(t_i-T)\right]\underline{\delta\phi}(t_i)\\
&= -\left\{\left[\underline{e}_{jn}^{T}(t_i)-\underline{e}_{jn}^{T}(t_i-T)\right]\underline{\underline{R}}_{An}(t_i)+\underline{e}_{jn}^{T}(t_i-T)\left[\underline{\underline{R}}_{An}(t_i)-\underline{\underline{R}}_{An}(t_i-T)\right]\right\}\underline{\delta\phi}(t_i)\\
&= -\left[\Delta\underline{e}_{jn}^{T}(t_i)\underline{\underline{R}}_{An}(t_i)+\underline{e}_{jn}^{T}(t_i-T)\Delta\underline{\underline{R}}_{An}(t_i)\right]\underline{\delta\phi}(t_i)\,. \qquad (82)
\end{aligned}
$$

The third term of (80) requires the dynamics matrix. The needed terms in $\underline{\underline{F}}[t_i;\underline{x}_n(t_i)]$ are simply understood in that the time derivative of position errors is equal to the velocity errors for the frame under consideration. Again, we have the same results as the corresponding term in (25) except here the time of evaluation is given.

$$\underline{e}_{jn}^{T}(t_i-T)\underline{\underline{K}}_{p}\underline{\underline{F}}[t_i;\underline{x}_n(t_i)]T\delta\underline{x}(t_i) = \underline{e}_{jn}^{T}(t_i-T)T\delta\underline{v}_u(t_i)\,. \qquad (83)$$

The fourth term of (80) results in coupling to gyro related errors that may be included in $\delta\underline{x}$. If all of these errors were lumped into a single angular velocity error of the body frame relative to an inertial frame, it should be evident that the results for (80) and (25) would be the same except again for the time of evaluation being explicitly known here. In this case,

$$\underline{e}_{jn}^{T}(t_i-T)\underline{\underline{R}}_{An}(t_i-T)\underline{\underline{K}}_{\phi}\underline{\underline{F}}[t_i;\underline{x}_n(t_i)]T \rightarrow \underline{e}_{jn}^{T}(t_i-T)\underline{\underline{R}}_{An}(t_i-T)T\delta\underline{\omega}_{ib}\,. \qquad (84)$$

The last term of (80) yields results identical to those in (25), the part of the dynamics matrix required here being quite simple in that the rate of change of user clock offset is equal to the user clock drift.

$$\underline{k}_{B}^{T}\underline{\underline{F}}[t_i;\underline{x}_n(t_i)]T\delta\underline{x}(t_i) = T\delta f_u\,. \qquad (85)$$

We have then compared the results of the second method to that of the first (for a delta range measurement residual) when a first order model is considered. Each case resulted in five terms (when we lump all gyro errors into a single angular velocity error) and the five terms have been shown to be equal except that the time of evaluation is quite clear for the second method. This overcomes

the major weakness in the first approach. The second method, as previously noted, increases uncertainty in the measurement model for any inadequacies in the dynamics model used. Both methods may be extended beyond a first order model.

The mathematical development for the second method has been somewhat more concise than that for the initial method. We have been able to keep track of time arguments, making a clear distinction between the continuous argument "t" and discrete measurement times "t_i". In addition, those quantities evaluated "on the nominal trajectory" have been identified and tracked throughout this development.

4.0 SIMPLIFIED ANALYSIS OF THE DELTA-RANGE MEASUREMENT FOR A SPINNING BODY

4.1 Introduction

This section provides a mathematical analysis of attitude estimation addressing specifically a spinning body scenario. As a result it facilitates gaining certain insights to and identifying sensitivities in spinning body applications. The analysis is limited to the case of an initial (constant) attitude error in a rotating but not translating vehicle. The results reveal the sensitivity to spin rate, DR integration time, lever arm length, and orientation of the satellite. The results also include an upper bound of the accuracy with which attitude errors can be estimated.

4.2 Analysis Of Delta-Range Measurements

Neglecting the effects of clock bias, atmospheric delays, and noise, the doppler shift of the incoming GPS signal is, approximately,

$$w_{DOP_j}(t) = \frac{2\pi f_{RF}}{c} \frac{d}{dt} \| \underline{r}_j(t) \| , \tag{86}$$

where f_{RF} is the GPS signal carrier frequency, $\underline{r}_j$ is the Line of Sight (LOS) vector (see figure 1), c is the speed of light, and the double lines denote magnitude of a vector quantity.

The delta-range measurement is physically obtained by sensing and integrating the doppler shift in (86) over an interval of time T.

$$z_{DR_j}(t_i) = \frac{c}{2\pi f_{RF}} \int_{t_i-T}^{t_i} w_{DOP_j}(\tau)\, d\tau . \tag{87}$$

Using (86) and the fundamental theorem of calculus in (87) we obtain

$$\begin{aligned} z_{DR_j}(t_i) &= \frac{c}{2\pi f_{RF}} \int_{t_i-T}^{t_i} \frac{2\pi f_{RF}}{c} \frac{d}{d\tau} \| \underline{r}_j(\tau) \| d\tau \\ &= \| \underline{r}_j(t_i) \| - \| \underline{r}_j(t_i - T) \| . \end{aligned} \tag{88}$$

Without loss of generality we can assume an inertial frame XYZ as shown in figure 2. For simplicity we assume no motion other than a constant spin ω about the Y-axis. Suppose the antenna is

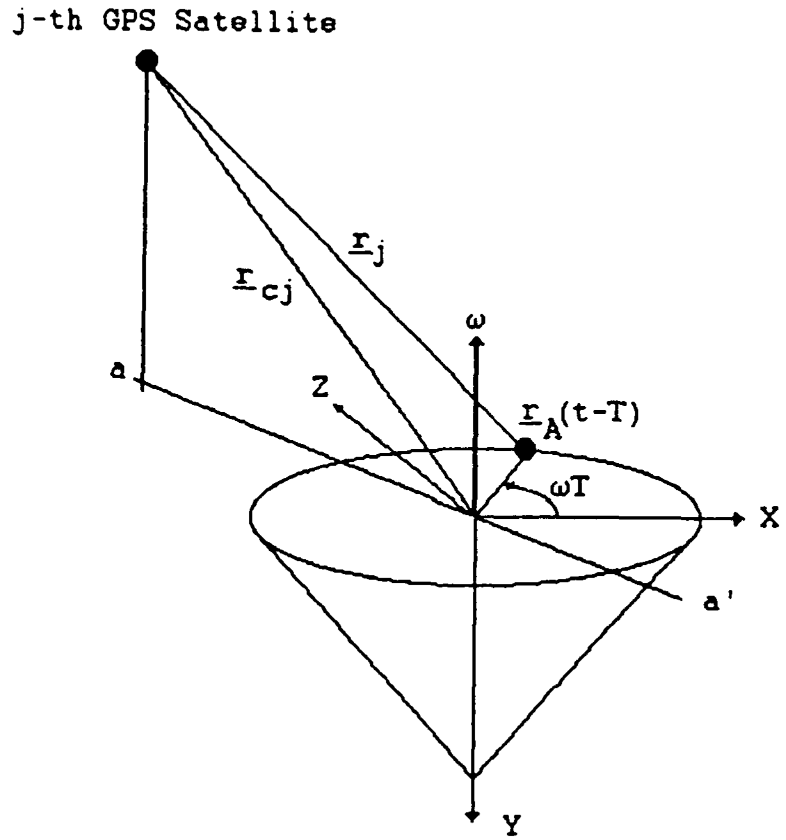


Figure 2. Geometry of simplified example.

initially located on the X-axis. Then, the lever-arm vector $\underline{r}_A$ is represented in the XYZ frame by

$$\underline{r}_A(t_i) = \begin{bmatrix} L\cos(\omega t_i) \\ 0 \\ L\sin(\omega t_i) \end{bmatrix}, \tag{89}$$

where L is the magnitude (length) of $\underline{r}_A$. The Line of Sight (LOS) range vector from the j-th satellite to the antenna at time t_i is given by (see Figure 1)

$$\underline{r}_j(t_i) = \underline{r}_{cj}(t_i) + \underline{r}_A(t_i). \tag{90}$$

In practice, T is sufficiently small so that in the interval $(t_i, t_i\text{-}T)$ the vector $\underline{r}_{cj}$ can be considered a constant, i.e., we can neglect the effect of satellite motion.

For the purpose of this analysis, let us consider the LOS vector as a function of the continuous time variable t as opposed to the discrete time t_i. Then

$$\|\underline{r}_j(t)\| = [(r_{cjx}+L\cos\omega t)^2 + (r_{cjy}-0)^2 + (r_{cjz}+L\sin\omega t)^2]^{1/2}. \tag{91}$$

Then

$$\frac{d}{dt}\|\underline{r}_j(t)\| = \frac{[(r_{cjx}+L\cos\omega t)(-L\omega\sin\omega t) + (r_{cjz}+L\sin\omega t)(L\omega\cos\omega t)]}{\|\underline{r}_j\|}. \tag{92}$$

Typically, the lever arm-length L is small compared to distance between the vehicle and a GPS satellite. Then

$$|r_{cjx}| >> |L\cos\omega t| \tag{93}$$

$$|r_{cjz}| >> |L\sin\omega t|. \tag{94}$$

Since L is assumed to be small, by taking

$$\|\underline{r}_j(t)\| \approx \|\underline{r}_{cj}\| \tag{95}$$

for any time t and neglecting terms in L^2, (92) reduces to

$$\frac{d}{dt}\|\underline{r}_j(t)\| = \frac{-r_{cjx}L\omega\sin\omega t + r_{cjz}L\omega\cos\omega t}{\|r_{cj}\|} \tag{96}$$

$$= L\omega\,[-e_{cjx}\sin\omega t + e_{cjz}\cos\omega t]\,,$$

where e_{cjx} and e_{cjz} are the X and Z components of a unit vector $\underline{e}_{cj}$ in the direction of $\underline{r}_{cj}$. Then from (86)

$$w_{DOFj}(t) = \frac{2\pi f_{RF}}{c}\,L\omega\,[-e_{cjx}\sin\omega t + e_{cjz}\cos\omega t]. \tag{97}$$

From (97), we conclude that the doppler shift is sinusoidal in time. The doppler shift zero crossings occur whenever

$$\tan(\omega t) = \frac{e_{cjz}}{e_{cjx}}\,, \tag{98}$$

i.e., whenever the antenna crosses the projection of $\underline{r}_{cj}$ onto the X-Z plane (indicated as line aa' in figure 2).

From (87) and (97) we obtain

$$z_{DRj}(t) = 2L\sin\left(\frac{\omega T}{2}\right)\left[-e_{cjx}\sin\left(\omega t - \frac{\omega T}{2}\right) + e_{cjz}\cos\left(\omega t - \frac{\omega T}{2}\right)\right]. \tag{99}$$

(99) can also be written in terms of the inner product between the vectors $\underline{e}_{cj}$ and $\underline{u}$ where

$$\underline{u} = \begin{bmatrix} -\sin\left(\omega t - \frac{\omega T}{2}\right) \\ 0 \\ \cos\left(\omega t - \frac{\omega T}{2}\right) \end{bmatrix}. \tag{100}$$

The vector $\underline{u}$ is orthogonal to the vector denoting the antenna location at the midpoint of the delta-range integration; i.e., $\underline{u}$ is orthogonal to

Therefore, for a constant rotation rate ω, $\underline{u}$ is parallel to the vector $\Delta\underline{r}_A$, which denotes the difference in antenna location at the start and end of the delta-range integration.

$$\underline{r}_A(t-\frac{T}{2}) = \begin{bmatrix} L\cos(\omega t-\frac{\omega T}{2}) \\ 0 \\ L\sin(\omega t-\frac{\omega T}{2}) \end{bmatrix} \tag{101}$$

$$\Delta\underline{r}_A(t) = \underline{r}_A(t) - \underline{r}_A(t-T). \tag{102}$$

Thus (99) can be rewritten as

$$z_{DRj}(t) = 2L\sin(\frac{\omega T}{2})\cos(\alpha[\underline{e}_{cj},\Delta\underline{r}_A(t)]), \tag{103}$$

where $\alpha[\underline{e}_{cj}, \Delta\underline{r}_A(t)]$ is the angle between the unit vector $\underline{e}_{cj}$ and $\Delta\underline{r}_A$. As indicated, this angle is a function of $\underline{e}_{cj}$ and $\Delta\underline{r}_A(t)$.

The following observations can now be made:

1) From (99), the delta-range measurement is sinusoidal in time. The delta-range measurement zero crossings occur whenever

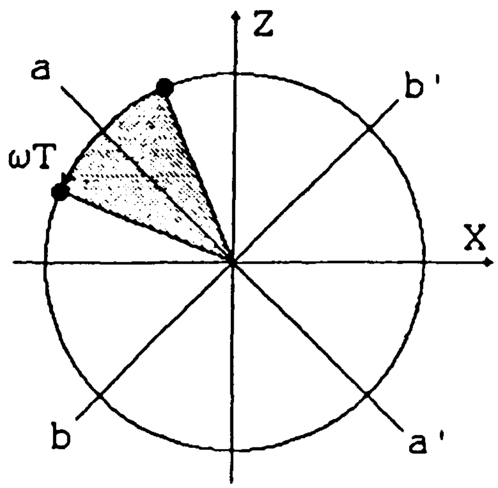
$$\tan(\omega(t-\frac{T}{2})) = \frac{e_{cjz}}{e_{cjx}}, \tag{104}$$

i.e., whenever the angle swept by the antenna lever arm during the integration time T is symmetrically distributed about the line aa'. This condition is illustrated in figure 3a.

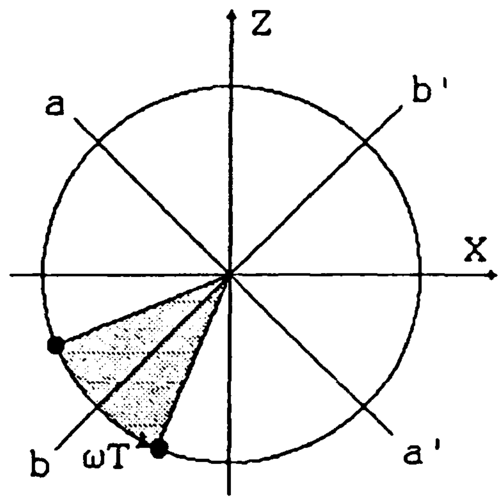
2) Setting the time derivative of (99) equal to zero we conclude that the peak values of the delta-range measurement occur whenever

$$\tan(\omega(t-\frac{T}{2})) = -\frac{e_{cjx}}{e_{cjz}}, \tag{105}$$

i.e., whenever the angle swept by the antenna lever arm during the integration time T is symmetrically distributed about the line bb' which is orthogonal to the projection of $\underline{r}_{cj}$ on the X-Z plane. This condition is illustrated in figure 3b.



a) Zero DR measurement and peak attitude error observability



b) Peak DR measurement and zero attitude error observability

Figure 3. Illustration of conditions for zero and peak delta-range measurement and attitude error observability.

3) From (103) we conclude that the maximum peak value of the delta-range measurement is 2L. This value is obtained when the vectors $\underline{e}_j$ and $\Delta\underline{r}_A$ are parallel and the antenna rotates 180 degrees starting and ending on the line aa'. On the other hand, the delta-range measurement is zero if the vectors $\underline{e}_j$ and $\Delta\underline{r}_A$ are orthogonal.

4.3 Analysis Of Attitude Error Observability

In this paragraph we investigate the observability afforded to attitude errors by the delta-range measurement. Attitude errors are directly observable whenever the corresponding entries in the observation matrix H are non zero - i.e., the entries in column $\delta\phi$ of (32). For "good" observability, these entries should be as large as possible so that the contribution of attitude errors to the DR residual is significant.

We now return to a discrete time formulation. From (32), the attitude error $\underline{\delta\phi}$ is related to the delta-range residual for the j-th satellite via

$$\underline{H}_{j\delta\phi} = -(\Delta\underline{e}_{jn}^T \underline{\underline{R}}_{An} + \underline{e}_{jn}^T \Delta\underline{\underline{R}}_{An}). \tag{106}$$

For our choice of reference frame and short periods of time the change in the unit vector $\underline{e}_{jn}$ is small so, we can assume that $\Delta\underline{e}_{jn}$ is approximately zero. Then (106) reduces to

$$\underline{H}_{j\delta\phi} \approx -\underline{e}_{jn}^T \Delta\underline{\underline{R}}_{An}. \tag{107}$$

The contribution of the attitude error $\underline{\delta\phi}$ to the DR measurement residual is obtained by multiplying both sides of (106) by $\underline{\delta\phi}$.

$$\underline{H}_{j\delta\phi}\underline{\delta\phi} = -\underline{e}_{jn}^T \Delta\underline{\underline{R}}_{An}\underline{\delta\phi} \tag{108}$$

$$-\underline{e}_{jn}^T \Delta\underline{\underline{R}}_{An}\underline{\delta\phi} \leftrightarrow -\overline{e}_{jn}^T \cdot \{[\overline{r}_A(t_i) - \overline{r}_A(t_i - T)] \times \overline{\delta\phi}\} \tag{109}$$

or

$$\underline{e}_{jn}^T \Delta\underline{\underline{R}}_{An}\underline{\delta\phi} \leftrightarrow -\overline{e}_{jn}^T \cdot \{[\Delta\overline{r}_A(T)] \times \overline{\delta\phi}\}. \tag{110}$$

From (110), we conclude that the contribution of the attitude error $\underline{\delta\phi}$ to the delta-range residual is maximum when the vectors $\underline{e}_{jn}$, $\Delta\underline{r}_A$, and $\underline{\delta\phi}$ are mutually orthogonal. The contribution is zero if any two are in the same plane.

From figure 2 we conclude that the skew-symmetric matrix R_{An} is

$$\underline{\underline{R}}_{An} = \begin{bmatrix} 0 & -L\sin\omega t_i & 0 \\ L\sin\omega t_i & 0 & -L\cos\omega t_i \\ 0 & L\cos\omega t_i & 0 \end{bmatrix}. \qquad (111)$$

Then

$$\Delta\underline{\underline{R}}_{An} = \begin{bmatrix} 0 & -L(\sin\omega t_i - \sin\omega(t_i - T)) & 0 \\ -\text{sym} & 0 & -L(\cos\omega t_i - \cos\omega(t_i - T)) \\ 0 & -\text{sym} & 0 \end{bmatrix}, \qquad (112)$$

where "sym" denotes an entry equal to the symmetric entry in the matrix. Using trigonometric identities (112) reduces to

$$\Delta\underline{\underline{R}}_{An} = \begin{bmatrix} 0 & -2L\sin\frac{\omega T}{2}\cos\omega(t_i - \frac{T}{2}) & 0 \\ -\text{sym} & 0 & 2L\sin\frac{\omega T}{2}\sin\omega(t_i - \frac{T}{2}) \\ 0 & -\text{sym} & 0 \end{bmatrix}. \qquad (113)$$

Since we have assumed that L is small compared to $\underline{r}_j$, $\underline{e}_{cj}$ is approximately equal to $\underline{e}_j$. Finally, assuming we are only interested in the y-component of the attitude error, we obtain from (107),

$$h_{j\delta\phi_y} = 2L\,\sin(\frac{\omega T}{2})\,[e_{cjx}\cos\omega(t_i - \frac{T}{2}) + e_{cjz}\sin\omega(t_i - \frac{T}{2})]. \qquad (114)$$

From (114) we conclude that $h_{j\delta\phi_y}$ is zero if any of the following three conditions occur:

$$L = 0, \qquad (115)$$

$$\frac{\omega T}{2} = N\pi \quad (N = 0,1,2,\ldots), \qquad (116)$$

$$e_{cjx}\cos(\omega t_i - \frac{\omega T}{2}) + e_{cjz}\sin(\omega t_i - \frac{\omega T}{2}) = 0. \qquad (117)$$

Obviously, the condition in (115) can occur if and only if the lever arm is zero. The condition in (116) can occur if there is no rotation or if the rotation rate is such that a multiple of 360 degrees is swept during the delta-range integration time T.

The condition in (117) can occur if

$$e_{cjx} = e_{cjz} = 0 \tag{118}$$

or

$$\tan(\omega t_i - \frac{\omega T}{2}) = -\frac{e_{cjx}}{e_{cjz}} . \tag{119}$$

The condition in (118) occurs if the satellite is directly above (or below) the vehicle; i.e., if the vector $\underline{r}_{cj}$ in figure 2 has only a Y component. The condition in (119) occurs if the angle swept by the antenna lever arm during the delta-range integration interval is symmetrically distributed about the line bb' in the X-Z plane - see Figure 3.

We can express (114) in terms of the inner product between $\underline{e}_{cj}$ and the antenna location vector at the midpoint of the delta-range integration, $\underline{r}_A(t_i - T/2)$.

$$\begin{aligned} h_{j\delta\phi_y} &= 2\sin(\frac{\omega T}{2})\,[\underline{e}_{cj}^T\,\underline{r}_A(t_i - \frac{T}{2})] \\ &= 2\,L\,\sin(\frac{\omega T}{2})\cos(\beta[\underline{e}_{cj}, \underline{r}_A(t_i - \frac{T}{2})]), \end{aligned} \tag{120}$$

where $\beta[\underline{e}_{cj}, \underline{r}_A(t_i - T/2)]$ is the angle between the unit vector $\underline{e}_{cj}$ and the antenna location vector $\underline{r}_A(t_i - T/2)$.

From (120) we conclude that $h_{j\delta\phi_y}$ is sinusoidal and has a maximum peak value of 2L. This value is obtained when the vectors $\underline{e}_{cj}$ and $\underline{r}_A(t_i - T/2)$ are parallel and the antenna rotates 180 degrees starting and ending on line bb'. $h_{j\delta\phi_y}$ is zero if the vectors $\underline{e}_{cj}$ and $\underline{r}_A(t_i - T/2)$ are orthogonal. It follows that maximum attitude error observability is obtained when the DR measurement is at a zero crossing.

4.4 Attitude Estimation Error Bound

Suppose an Information Kalman Filter is used to estimate a constant attitude error $\delta\phi_y$ using DR measurements. The variance of the estimation error at t_k is (Anderson & Moore, 1979)

$$(\sigma^2_{\varepsilon\delta\phi_y}(t_k))^{-1} = (\sigma^2_{\varepsilon\delta\phi_y}(t_0))^{-1} + \sum_{i=0}^{k} \frac{h_{j\delta\phi_y}(t_i)^2}{\sigma^2_{DR}} , \qquad (121)$$

where σ_{DR} denotes the sigma value of the observation noise.

Numerical values for $h_{j\delta\phi y}(t_i)$ are required in order to evaluate the sum in (121) . Assuming that t_k is sufficiently large, we can assume that the sum can be evaluated using an average (and therefore constant) value in place of $h_{j\delta\phi y}(t_i)$; i.e.,

$$\sum_{i=0}^{k} \frac{h_{j\delta\phi_y}(t_i)^2}{\sigma^2_{DR}} \approx (k+1) \frac{[\mathrm{Avg}(h_{j\delta\phi_y})]^2}{\sigma^2_{DR}} . \qquad (122)$$

Assuming that to a sufficient approximation

$$\mathrm{Avg}(h_{j\delta\phi_y}) \approx \frac{\mathrm{Peak}(h_{j\delta\phi_y})}{2} , \qquad (123)$$

(121) reduces to

$$\begin{aligned} (\sigma^2_{\varepsilon\delta\phi_y}(t_k))^{-1} &= (\sigma^2_{\varepsilon\delta\phi_y}(t_0))^{-1} + (k+1) \frac{\mathrm{Peak}(h_{j\delta\phi_y})^2}{4\sigma^2_{DR}} \\ &= (\sigma^2_{\varepsilon\delta\phi_y}(t_{k-1}))^{-1} + \frac{\mathrm{Peak}(h_{j\delta\phi_y})^2}{4\sigma^2_{DR}} \qquad (124) \\ &\geq \frac{\mathrm{Peak}(h_{j\delta\phi_y})^2}{4\sigma^2_{DR}} . \end{aligned}$$

Using (120) in (124) and taking the square root of both sides we obtain

$$\sigma_{\varepsilon\delta\phi_y}(t_k) \leq \frac{\sigma_{DR}}{L \sin(\frac{\omega T}{2}) \cos\beta} \qquad (125)$$

for $t_k >> 0$.

The following conclusions can be drawn:

1) The uncertainty in estimating attitude errors $\sigma_{\epsilon\delta\phi}$ with an offset antenna is proportional to the delta-range measurement noise, i.e., σ_{DR}. (σ_{DR} is inversely proportional to the SNR at the input of the carrier tracking loop).

2) $\sigma_{\epsilon\delta\phi}$ is inversely proportional to the length of the antenna lever arm L.

3) $\sigma_{\epsilon\delta\phi}$ is minimized if the angle swept during the delta-range integration is 180 degrees.

4) $\sigma_{\epsilon\delta\phi}$ is minimized if the satellite is in the plane defined by the rotating antenna and the angle swept during the delta-range integration is symmetrically distributed about the vector r_{cj}.

We note that (125) is only a bound. As a result, it is not very useful in determining the uncertainty in the attitude estimates exactly. It is, however, useful in evaluating the sensitivity of the uncertainty to L and ωT. To illustrate this, let γ be a factor such that strict equality is achieved in (125) for some L_0, ω_0, and T_0; i.e.,

$$\sigma_{\epsilon\delta\phi_y}(t_k;L_0,\omega_0,T_0) = \frac{\gamma\,\sigma_{DR}}{L_0 \sin(\frac{\omega_0 T_0}{2}) \cos\beta}. \qquad (126)$$

Then, for different L, ω, and T we have

$$\sigma_{\epsilon\delta\phi_y}(t_k;L,\omega,T) = \frac{L_0}{L}\,\frac{\sin(\frac{\omega_0 T_0}{2})}{\sin(\frac{\omega T}{2})}\,\sigma_{\epsilon\delta\phi_y}(t_k;L_0,\omega_0,T_0). \qquad (127)$$

For example, increasing the lever arm from L = 5 inches to 20 inches, reduces the steady state attitude error bound by a factor of 4. On the other hand, with $\omega_0 = \omega = 450$ deg/sec, increasing the DR integration time from T = 0.78 sec to T = 1.0 sec, reduces the steady state attitude error bound by a factor of 9.

5.0 SIMULATION RESULTS

5.1 Introduction

This section provides covariance simulation analysis for a spinning body by using the more general results of section 3. The simulation scenario consists of a vehicle falling freely in an exoatmospheric trajectory. The results illustrate the ability of estimating attitude and attitude rate errors with an offset antenna. The sensitivity to the lever arm length and the spin rate were also investigated. To an approximation, the scenario here resembles the conditions assumed in the analysis of section 4. Consequently, the insights gained in section 4 carry over to the more realistic conditions of this section.

5.2 Scenario Description

We consider a vehicle spinning about its Y_b (body frame) axis at a constant roll rate of ω deg/sec. The vehicle is assumed to be equipped with a tightly integrated GPS/Inertial navigator mounted along the Y_b axis and near the vehicle's center of gravity. The GPS antenna is mounted on the perimeter of the vehicle at a distance L from the Y_b axis. The antenna therefore rotates at the vehicle spin rate ω. The vehicle moves along an exoatmospheric trajectory. The vehicle's altitude profile versus time is shown in figure 4.

Assuming a Y_b gyro scale factor error of 50 ppm, a spin rate of 450 deg/sec, and a scenario duration of 1,400 sec, an attitude error about the Y_b axis of 31.5 degrees is expected. The main objective of the simulations is to demonstrate the ability to estimate this attitude error by using the approach discussed in this report.

5.3 Simulation Program Description

The simulations were performed with Galaxy's Covariance Analysis Package (CAP) which is shown in figure 5. CAP includes a trajectory generator program (TRAGEN), a covariance analysis program (GGPSIM), a sensitivity analysis program (SENSIS), and various data reduction and plotting software packages. A short description of these programs is included in the following paragraphs.

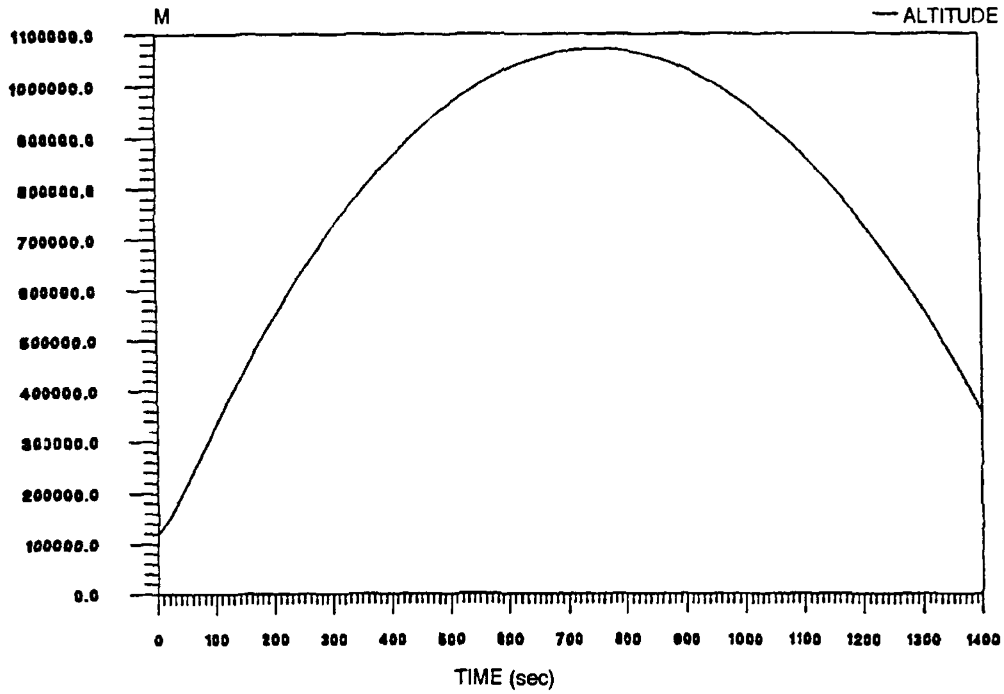


Figure 4. Altitude vs. time.

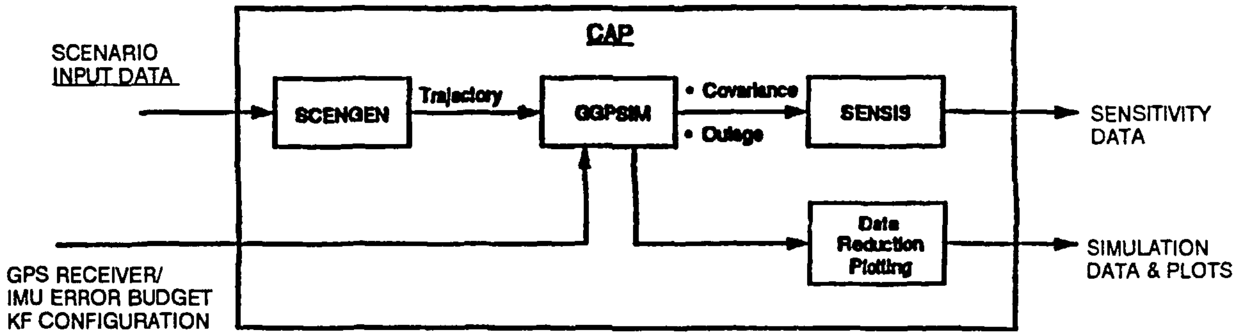


Figure 5. Covariance analysis package (CAP) block diagram.

5.3.1 Trajectory Generator (TRAGEN) Description

TRAGEN is a 6 Degree of Freedom (DOF), free falling body trajectory generator.

5.3.2 Covariance Simulator (GGPSIM) Description

GGPSIM is a covariance simulation program of a tightly integrated GPS/Inertial navigator. A top level block diagram is shown in figure 6.

GGPSIM contains a "truth" and a "filter" covariance model. The truth model represents all errors that would affect the navigator. The truth model consists of the 80 errors listed in table 2. Collectively, these errors comprise the true error state of the navigator. The "filter" model consists of a subset of the errors in the true error state. These errors comprise the navigator's Kalman Filter state vector, i.e., they represent the system's perception of the errors in the real world. The errors that are typically included in the filter state vector are the ones that can be unambiguously estimated (i.e., are fully observable) through GPS measurements. The maximum number of such states in GGPSIM is 56 and are listed in table 2. In practice, the number of errors included in the filter state vector is further limited due to processing constraints. GGPSIM allows the user to arbitrarily select the states in the filter. This feature allows analysis of any conceivable GGP Kalman Filter configuration

GGPSIM is initialized with the covariance of the 80-element true error state and the covariance of the filter state. These covariances constitute the true and GGP Kalman Filter "error budget". The truth and filter covariances are propagated forward in time according to the vehicle's trajectory until a set of GPS measurements is obtained. GGPSIM simulates a maximum of 10 GPS channels, supplying the filter with 9 pairs of pseudorange and delta-range measurements. (It is assumed that one channel is used for simultaneous ionospheric delay measurements, and to aid in acquisition.) Therefore, for the 10-channel receiver, only nine channels provide pseudorange and delta range measurements). The 10 satellites are selected according to first come first served from all in view. GGPSIM also allows simulation with a 5 channel GPS receiver (4 pairs of pseudorange and delta range measurements). In this case the satellites are selected according to GDOP. The measurements are computed from a 24 GPS Satellite Constellation model which is integrated with GGPSIM.

The measurements are processed with a Kalman Filter which updates the filter covariance at 1 Hz with Carlson's algorithm (Anderson & Moore, 1979; Pitman, 1962). The computed Kalman Filter gains are used in configuring (through augmentation) the gain for the truth model.

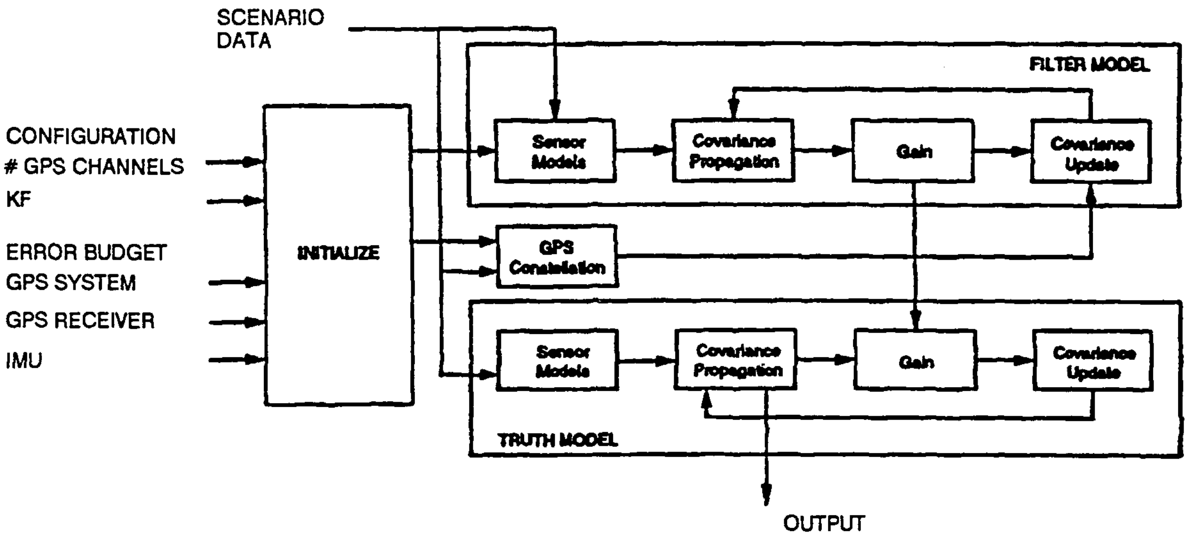


Figure 6. GGPSIM block diagram.

Table 2. GGPSIM true and filter error states.

STATE DESCRIPTION	TRUTH MODEL	FILTER(*)
Position Errors	3	3
Velocity Errors	3	3
IMU Attitude Errors	3	3
Gravity Deflections and Anomalies	3	3
Gyro Drift Rates	3	3
Gyro Input Axis G-Sensitivities	3	3
Gyro Spin Axis G-Sensitivities	3	3
Gyro Output-Axis G^2 Sensitivities	3	3
Gyro Scale Factor Errors	6	6
Gyro Input Axes Misalignments	6	6
Accelerometer Biases	3	3
Accelerometer Scale Factor Errors	3	3
Accelerometer Input Axes Misalignments	6	6
Altitude Sensor Error	1	1
Clock Errors:		
Phase Error	1	1
Frequency Error	1	1
Aging Error	1	1
Random Frequency Error	1	1
Acceleration Sensitivities	3	3
Satellite Ephemeris and Clock Residual Ranging Errors	24	0
TOTAL NUMBER OF STATES	80	56

(*) Maximum Number of States

The truth model's covariance is updated with the Joseph form. The covariances are then propagated forward in time using trajectory data until the next set of GPS measurements is obtained.

Two GGPSIM features are worth noting. First, the filter gains are computed using only the filter covariance elements. This feature enables GGPSIM to simulate realizable Kalman Filters. Second, GGPSIM automatically adjusts the filter input noise process covariance (typically denoted by Q) to account for the true error states that were not included in the filter state. This feature enables GGPSIM to simulate fine-tuned filters.

5.3.3 Sensitivity Analysis Program (SENSIS)

SENSIS is a simulation data post processing program that identifies the sensitivity and correlation of navigation errors (e.g., position errors, velocity errors, etc.) to inertial system errors (e.g., initial errors, alignment errors, biases, noise processes, etc.). In this analysis, SENSIS was used to identify correlations of attitude and attitude rate errors to other navigation errors.

5.4 Simulation Conditions And Assumptions

This paragraph describes the conditions, parameter values, and assumptions used in the simulations. The error budget for the GPS and the inertial errors is shown in table 3. These values are typical of advanced GPS receiver designs and navigation grade inertial components.

A short description and rationale for these conditions is given in the following paragraphs.

5.4.1 IMU Error Budget

A strapdown IMU with Fiber Optic Gyros (FOG) was assumed. Since FOGs exhibit very little g or g^2 sensitivity, these errors were assumed to be zero. Gyro and accelerometer biases and scale factors were chosen according to the requirements for the Phase 1 GPS Guidance Package (GGP). These values imply an inertial system of roughly 0.58 nmi/hr. The remaining IMU errors were assigned values that are typical of this class of inertial navigation system.

It is noted that the values assumed for the inertial components are <u>not</u> typical of inertial systems intended for space applications; for example, space applications generally require gyro scale factor errors of a few ppm (Pitman, 1962) as opposed to 50 ppm assumed here. The 50 ppm value was intentionally assumed here in order to illustrate the concept.

Table 3. Simulation error budget.

ERROR STATE		INITIAL 1σ VALUE	UNITS	ASSOCIATED NOISE INPUT	UNITS
Clock Phase		10^{-8}	sec	9.17×10^{-13}	sec/$\sqrt{\text{sec}}$
Clock Frequency		9×10^{-9}	sec/sec	N/A	
Clock Aging Error		$9.54(10)^{-7}$	sec/sec/yr	N/A	
Clock Random Freq. Error		9×10^{-9}	sec/sec	Correlation Time 5400	sec
Clock Accel. Sensitivities	X	8×10^{-10}	sec/sec/g	N/A	
	Y	4×10^{-11}	sec/sec/g	N/A	
	Z	2×10^{-9}	sec/sec/g	N/A	
Position (East)		0.4×10^{-3}	mrad	N/A	
Position (North)		0.4×10^{-3}	mrad	N/A	
Position (Vertical		5.5	m	N/A	
Velocity (East)		0.4×10^{-2}	m/sec	0.03	m/sec/$\sqrt{\text{hr}}$
Velocity (North)		0.4×10^{-2}	m/sec	0.03	m/sec/$\sqrt{\text{hr}}$
Velocity (Vertical)		0.8×10^{-2}	m/sec	0.03	m/sec/$\sqrt{\text{hr}}$
Attitude (East)		0.6×10^{-1}	mrad	0.005	deg/$\sqrt{\text{hr}}$
Attitude (North)		0.46×10^{-1}	mrad	0.005	deg/$\sqrt{\text{hr}}$
Attitude (Vertical)		0.4×10^{-1}	mrad	0.005	deg/$\sqrt{\text{hr}}$

Table 3. Simulation error budget. (Continued)

ERROR STATE		INITIAL 1σ VALUE	UNITS	ASSOCIATED NOISE INPUT	UNITS
Gyro Bias	X	0.86×10^{-2}	deg/hr	1.67×10^{-4}	deg/hr√s
	Y	0.99×10^{-2}		Correlation Time 7200	sec
	Z	0.86×10^{-2}			
Gyro Scale Factor	+X	48.6	ppm	N/A	
	-X	21.6	ppm	N/A	
	+Y	50	ppm	N/A	
	-Y	50	ppm	N/A	
	+Z	50	ppm	N/A	
	-Z	50	ppm	N/A	
Gyro Misalignment	X about Y	0.048	mrad	N/A	
	X about Z	0.048	mrad	N/A	
	Y about X	0.048	mrad	N/A	
	Y about Z	0.044	mrad	N/A	
	Z about X	0.048	mrad	N/A	
	Z about Y	0.047	mrad	N/A	
Gyro G, G^2- Sensitivities	(9)	0	deg/hr/g, deg/hr/g^2	0	
Accelerometer Bias	X	17	μg	0.833	μg√sec
	Y	20	μg	Correlation Time 7200	μg√sec
	Z	23	μg		μg√sec
Accelerometer Scale Factors	X	92	ppm	N/A	
	Y	51	ppm	N/A	
	Z	100	ppm	N/A	

Table 3. Simulation error budget. (Continued)

ERROR STATE		INITIAL 1σ VALUE	UNITS	ASSOCIATED NOISE INPUT	UNITS
Accelerometer Misalignments	X about Y	0.048	mrad	N/A	
	X about Z	0.04	mrad	N/A	
	Y about X	0.048	mrad	N/A	
	Y about Z	0.045	mrad	N/A	
	Z about X	0.043	mrad	N/A	
	Z about Y	0.047	mrad	N/A	
Gravity Deflection (East)		21.8	μg	Correlation Distance 10	nm
Gravity Deflection (North)		18.5	μg	Correlation Distance 10	nm
Gravity Deflection (Vertical)		34	μg	Correlation Distance 60	nm
Altitude Sensor Bias		N/A		N/A	
SV Ephemeris & Clock	(24)	3.88	m	Correlation Time 1.44×10^4	sec

5.4.2 GPS Error Budget

A 10-channel GPS receiver was assumed to establish an upper bound of performance capability. It was assumed that 9 channels tracked satellites (all in view) supplying the filter with 9 pairs of pseudorange and delta range measurements. The 10th channel was assumed to be providing ionospheric measurements.

The GPS receiver accuracies were chosen according to SS-US-200 (1979) and SS-GPS-300B (1980) and are similar to those of the GPS UE 3A receiver. These values were chosen as a worst case in accordance with the minimum requirements specified in the GGP System Specification.

The clock frequency accuracy was derived from SS-US-200 (1979). The remaining clock errors were assigned values which are typical of oscillators used in current GPS applications.

The following assumptions were made: a mask angle of 5° above the horizon; 3-db noise figure; omni directional antenna; antenna shadowing was neglected as a simplifying assumption; satellite ephemeris and clock residual errors expected of the Block II constellation were used.

As previously noted, the rotating antenna introduces a systematic phase shift to the received GPS signal. It is assumed that this phase shift is properly compensated for by the tightly integrated navigator. This is a reasonable assumption since the gyros are able to sense the rotation to within 50 ppm. The residual phase error is assumed to be is negligible as far as the GPS pseudo-range and delta-range measurements are concerned.

5.4.3 Kalman Filter Configuration

An "optimal" Kalman Filter configuration was used. The state vector included all filter states listed in table 2 except for gyro g and g^2 sensitivities (a total of 47 states). This configuration was used in order to identify which and to what extent the various error states can be estimated. In reality, reduced order filters will be used. In these realistic configurations, states that can not be estimated to a significant degree are removed from the state vector. This analysis, however, was not done here.

5.5 Simulation Results

Results were first obtained for three sets of lever arm length L, spin rate ω, and delta-range measurement integration time T. These results are discussed as Cases 1, 2, and 3 below. Additional results were then obtained to reveal the sensitivity of attitude errors to antenna-lever-arm length and spin rate.

5.5.1 Case 1: $L = 0.0$ inches, $\omega = 450$ deg/sec, $T = 0.78$ sec

This case was considered in order to establish a baseline. Physically, this case would result if the antenna were mounted on the Y_b axis - figure 1. As per the discussion in section 2, attitude error estimation with a zero lever arm is not possible.

Figure 7 shows the resulting Position Errors vs Time and figure 8 shows the resulting Velocity Errors vs Time. The results indicate good estimation of position and velocity errors due to the processing of GPS measurements. The residual position errors are due to satellite biases. The residual velocity errors are small (0.01 m/s) due to smooth dynamics, good satellite observability, and lack of jamming. The "bumps" in the velocity error profiles are due to satellite switching.

In typical terrestrial navigation applications, good position and velocity-error estimation would imply good attitude error estimation since attitude errors are coupled to velocity errors via the specific forces (Schuler Loop). In this case, however, the vehicle is under free-fall conditions and the specific forces are zero. Consequently, the attitude errors are decoupled from position and velocity errors and without a lever arm there is no mechanism for attitude estimation.

Due to the scale factor error, the attitude error about the Y_b axis is expected to grow. Indeed, the attitude error profiles in figure 9 (Geographic Coordinates) and figure 10 (RSS Attitude Error) show a roughly linear growth. The attitude error is shown to grow to approximately 30 degrees as predicted earlier.

In addition, none of the instrumentation errors affecting attitude (e.g., gyro scale factor errors, gyro bias, etc.) can be estimated. For example, figure 11 shows all gyro scale factors errors remain constant throughout the trajectory.

In essence, under the conditions of this case, GPS succeeds in estimating the translational errors (position and velocity) as if the vehicle were a point mass. There is no mechanism to estimate attitude errors either as direct observables or through coupling to other directly observable errors.

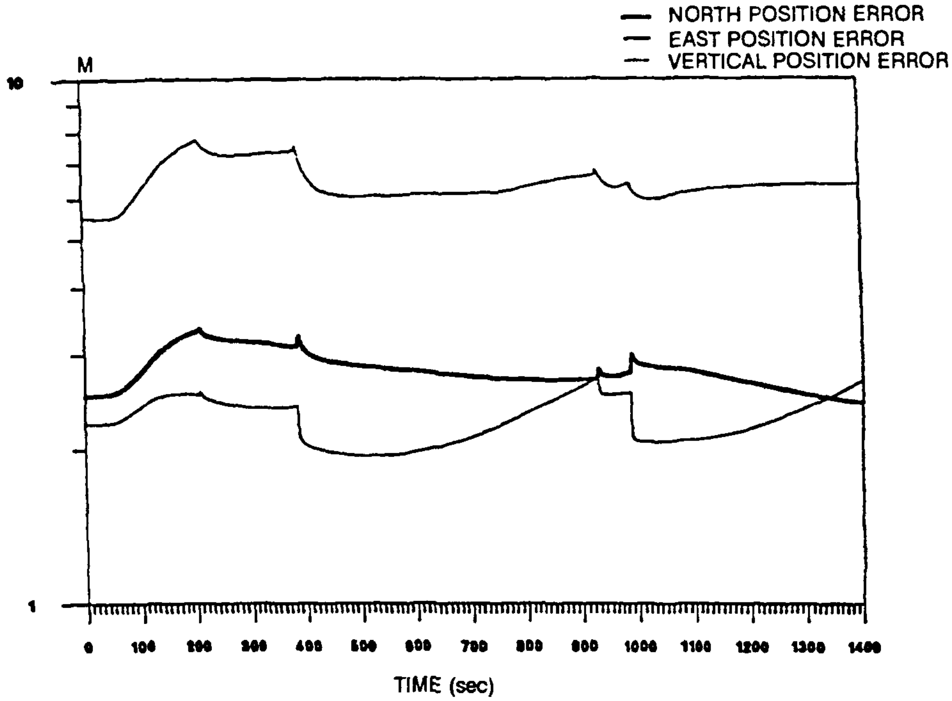


Figure 7. Case 1: position errors vs. time.

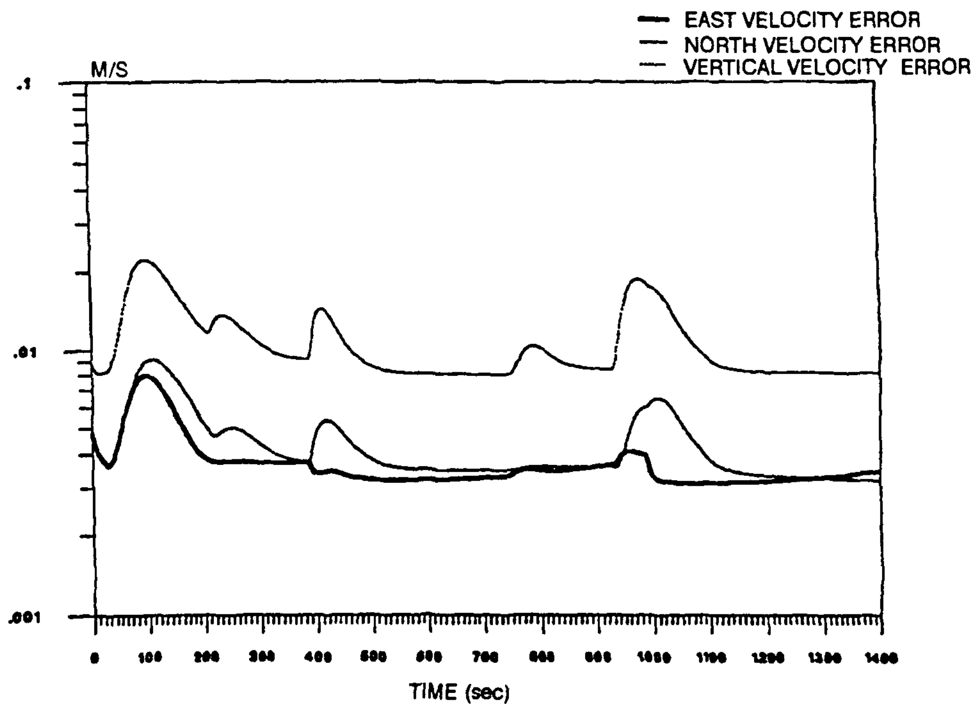


Figure 8. Case 1: velocity errors vs. time.

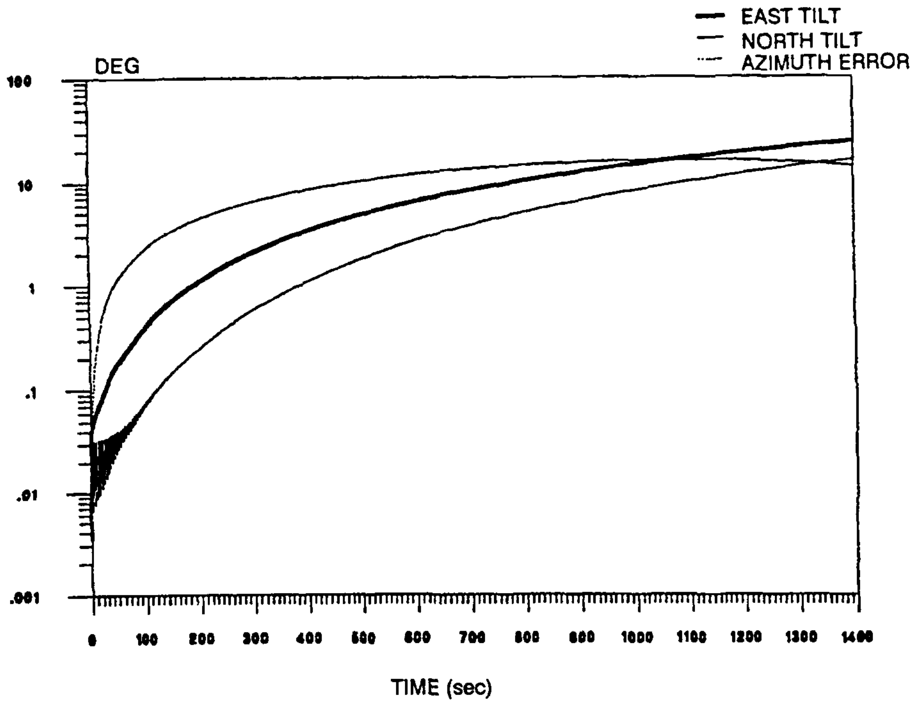


Figure 9. Case 1: attitude errors vs. time.

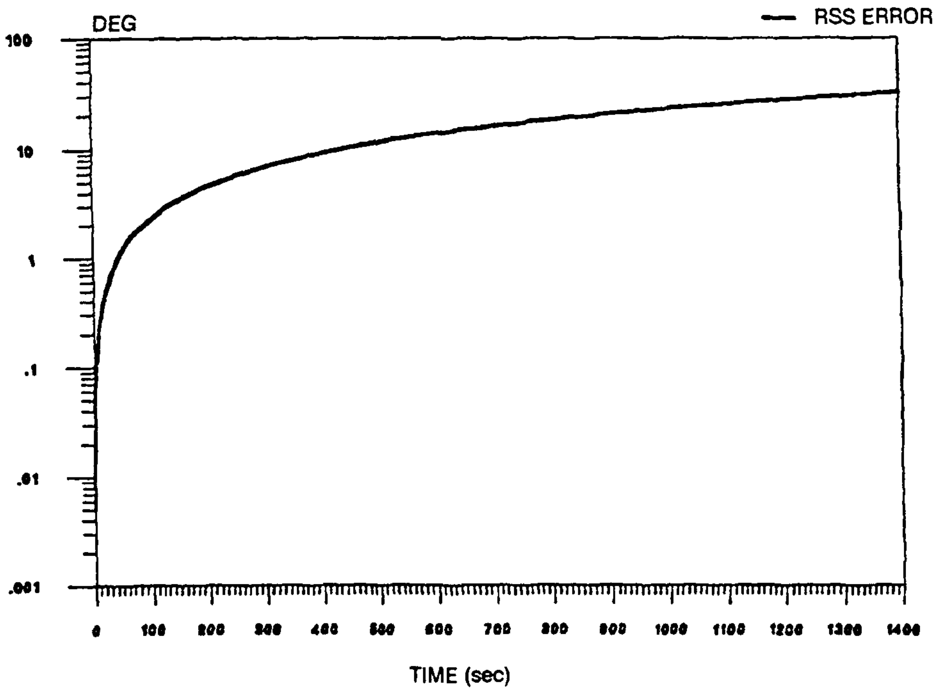


Figure 10. Case 1: RSS attitude errors vs. time.

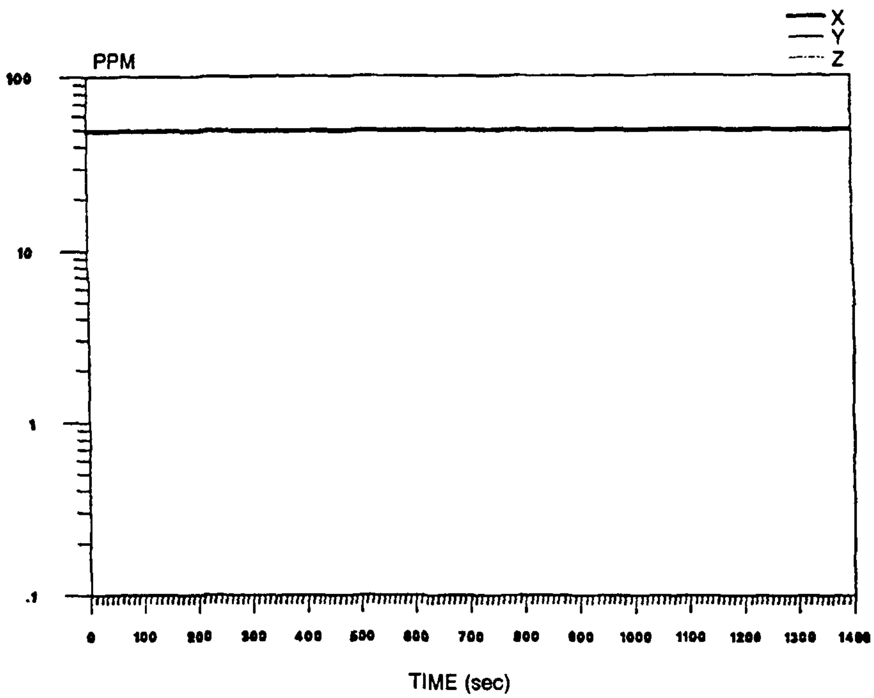


Figure 11. Case 1: gyro scale factor errors vs. time.

5.5.2 Case 2: $L = 7.0$ inches, $\omega = 450$ deg/sec, $T = 0.78$ sec

In this case, a lever arm of $L = 7$ inches as shown in figure 1 is assumed. Again, good position and velocity error performance is obtained (figures 12 and 13). In addition, however, attitude errors are estimated. Figures 14 and 15 illustrate the attitude error profiles in Geographic Coordinates and RSS value, respectively.

The results show that attitude error has been estimated to within 2 degrees - roughly, an order of magnitude improvement over Case 1. The RSS attitude error profile in figure 15 consists of a period of transient performance followed by a nearly flat behavior. This is due to the estimation of the instrumentation errors. Figure 16 shows that the Y_b axis gyro scale factor error is estimated to within TBD ppm. The other scale factor errors are not estimated since there is no rotation about those axes.

Note that the position and velocity error profiles here are identical to those in Case 1. This illustrates that position and velocity errors are fully decoupled from attitude errors under free fall conditions. Consequently, although here attitude errors were reduced through estimation, the position and velocity errors were not reduced any further.

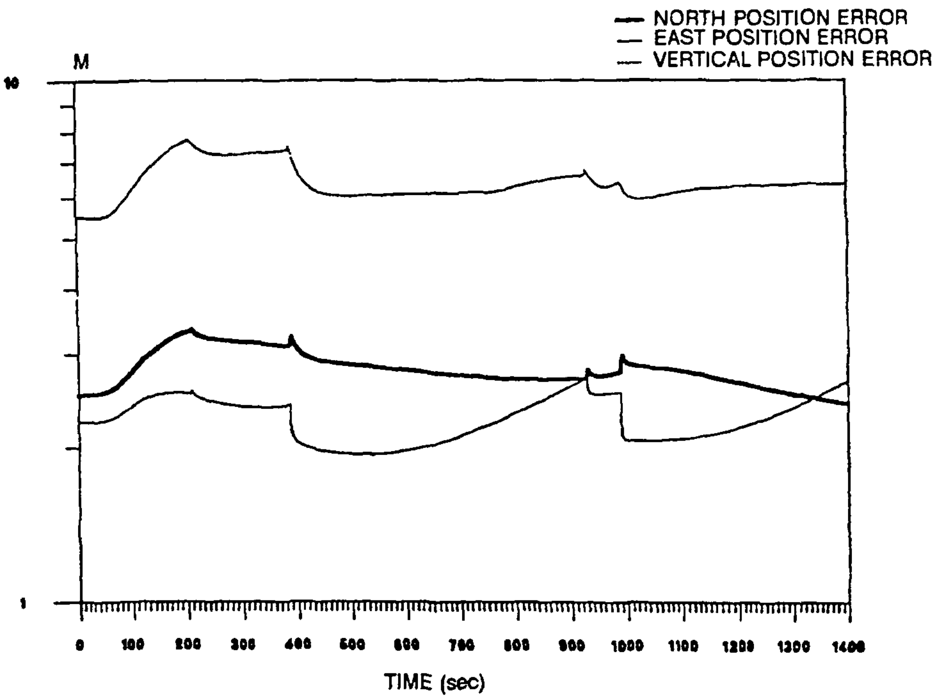


Figure 12. Case 2: position errors vs. time.

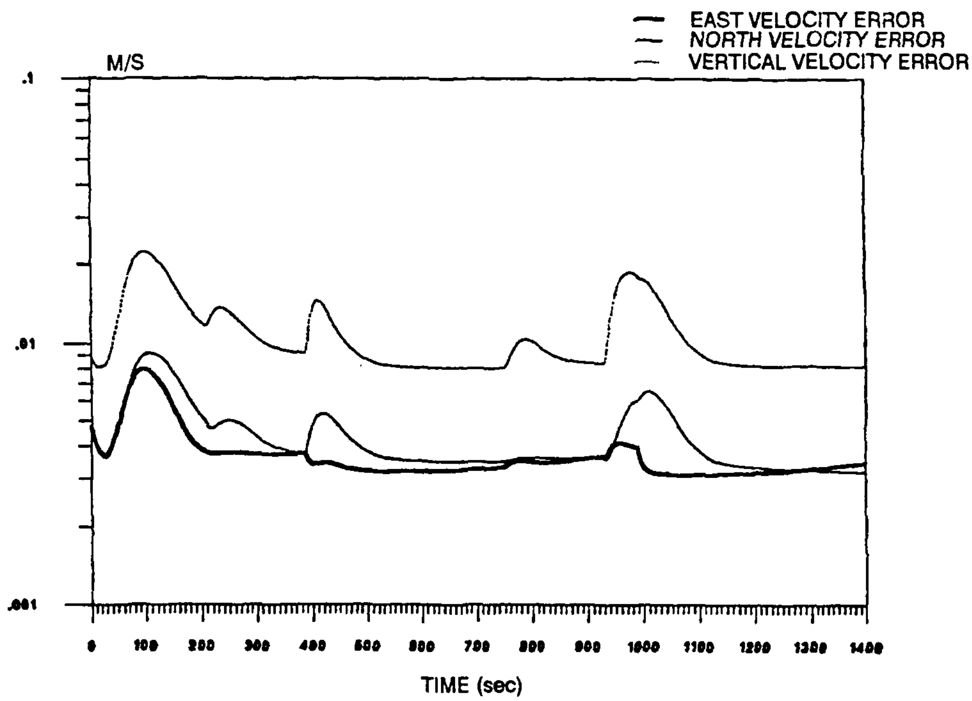


Figure 13. Case 2: velocity errors vs. time.

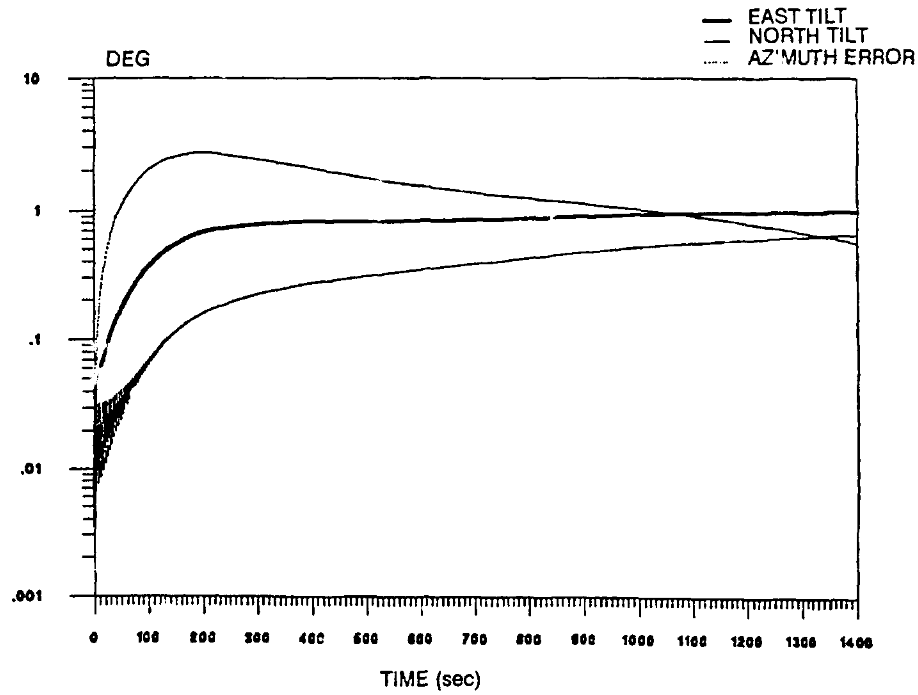


Figure 14. Case 2: attitude errors vs. time.

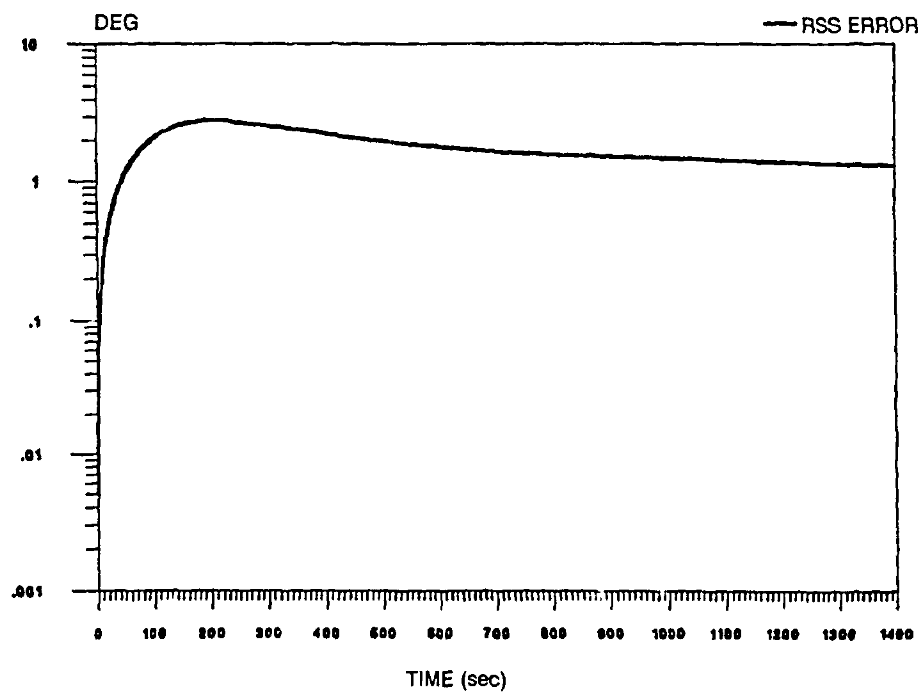


Figure 15. Case 2: RSS attitude errors vs. time.

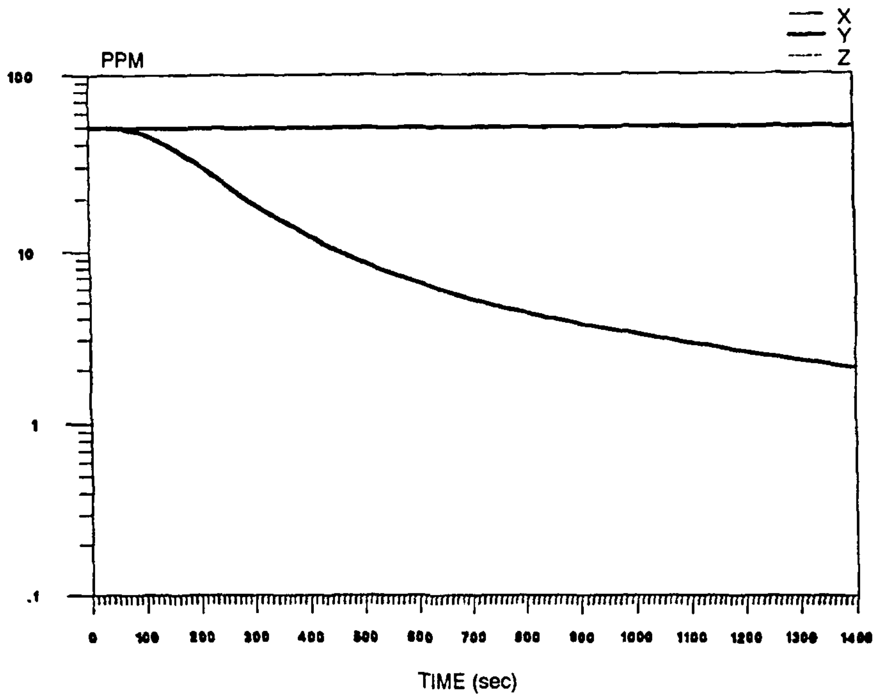


Figure 16. Case 2: gyro scale factor errors vs. time.

5.5.3 Case 3: L = 7.0 inches, ω = 450 deg/sec, T = 1.00 sec

In this case the delta-range integration time is increased to 1 sec. As per the discussion in Section 4, the increase in delta-range integration time results in a more favorable product ωT (angle swept during the DR integration time). Again, the position and velocity error profiles were identical to the ones obtained in the previous cases (figures 17 and 18).

The attitude error profiles are shown in figures 19 and 20 in Geographic Coordinates and RSS value, respectively. Figure 20 shows that the attitude error is now estimated to within 0.2 degrees, roughly two orders of magnitude improvement over Case 1 and an order of magnitude improvement over Case 2. Figure 21 shows that the scale factor error is now estimated to within 0.2 ppm.

5.5.4 Sensitivity Results

Additional simulations were performed using the lever arm length L and the spin rate ω as simulation parameters. Figure 22 shows the RSS attitude error at the end of the simulation vs the lever arm length. Figure 23 shows the RSS attitude error at the end of the simulation vs the spin rate. These results agree with the predictions in Section 4, i.e., the attitude error is inversely proportional to the lever-arm length and shows a cosecant dependence on the spin rate.

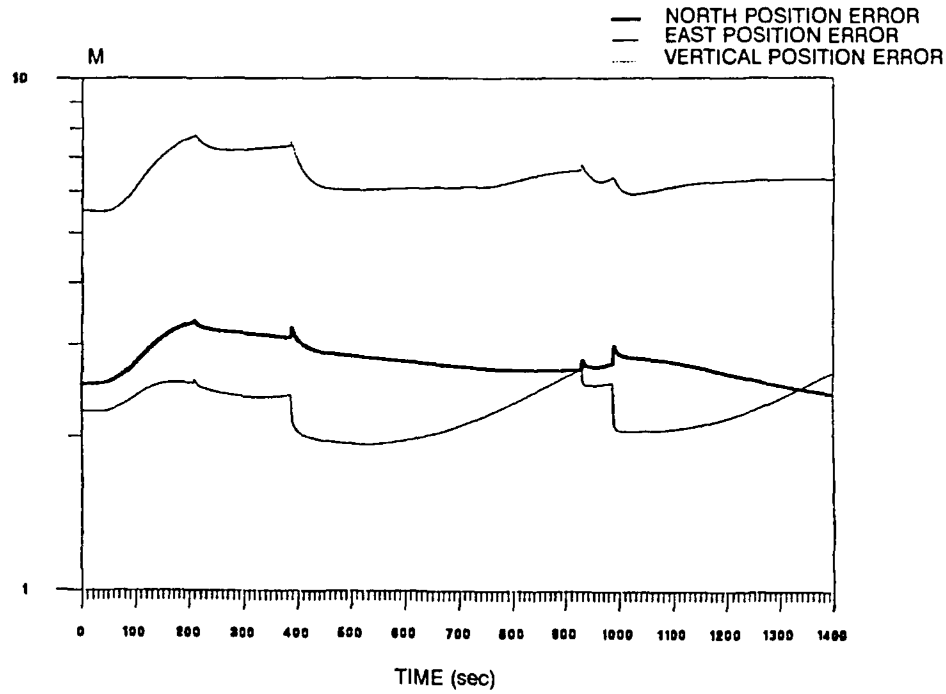


Figure 17. Case 3: position errors vs. time.

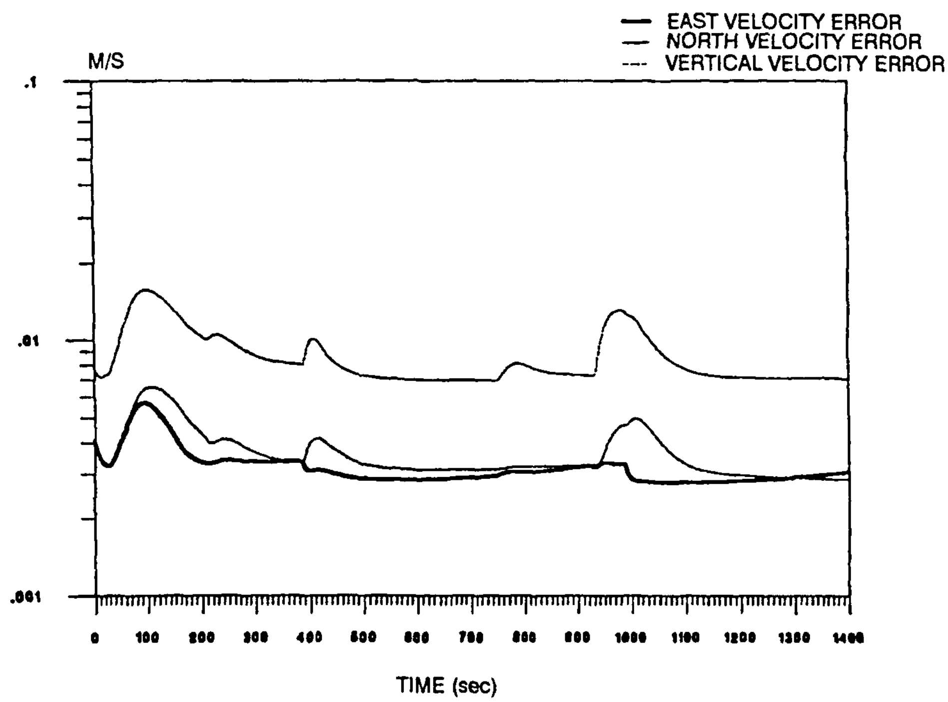


Figure 18. Case 3: velocity errors vs. time.

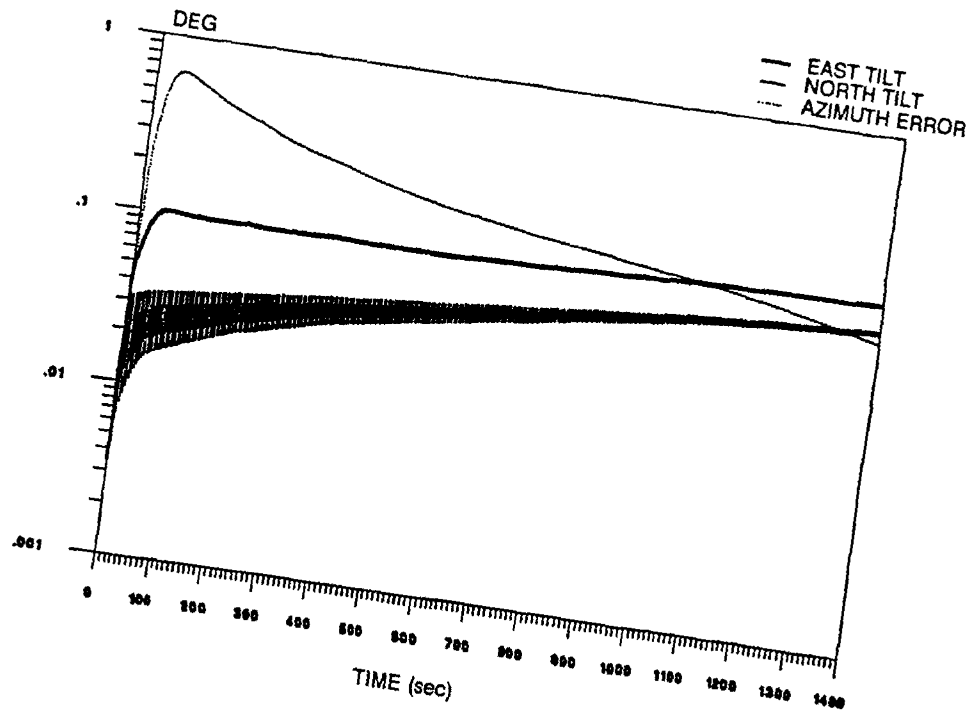


Figure 19. Case 3: attitude errors vs. time.

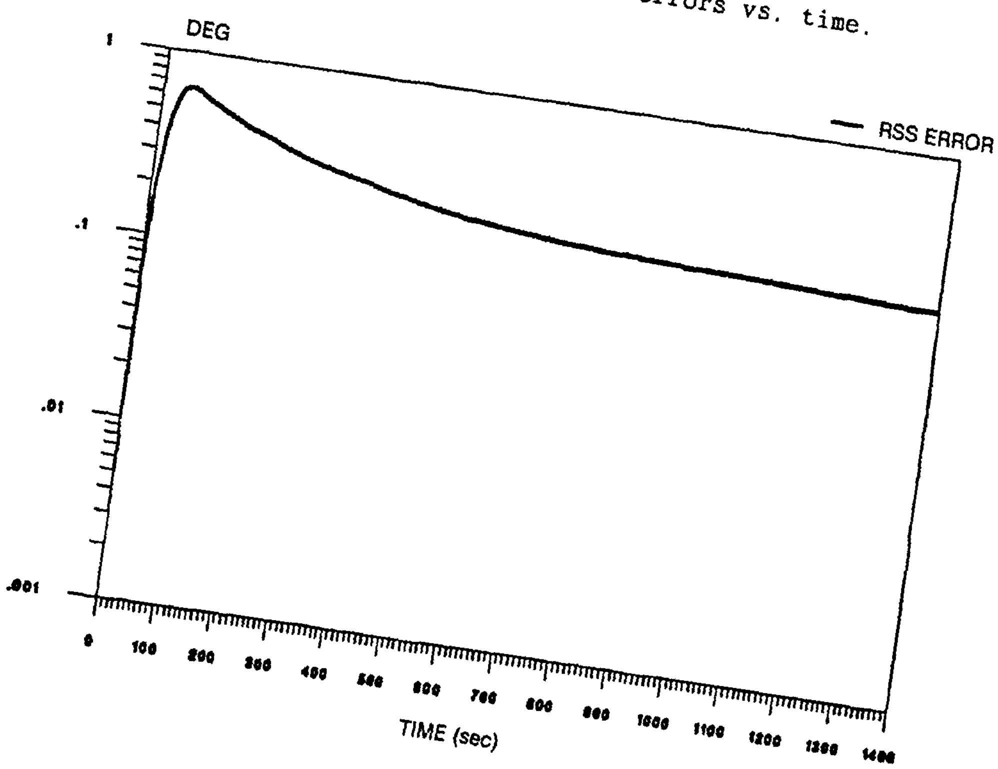


Figure 20. Case 3: RSS attitude errors vs. time.

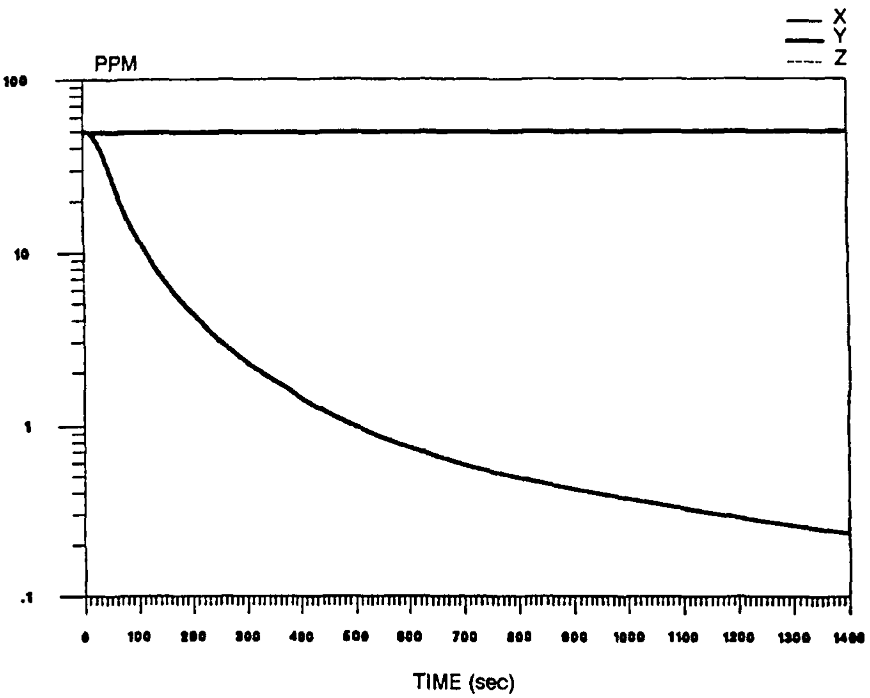


Figure 21. Case 3: gyro scale factor errors vs. time.

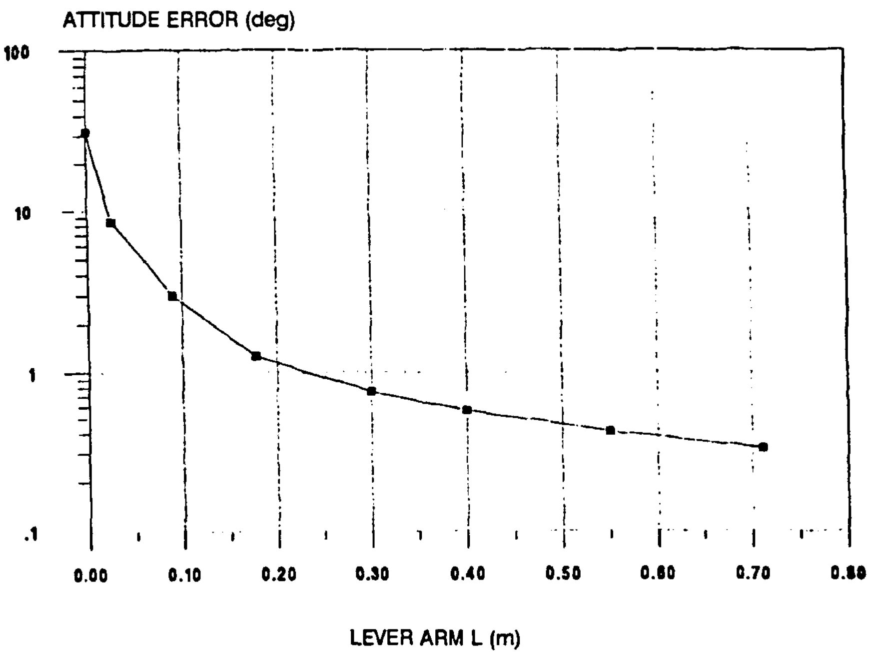


Figure 22. Final RSS attitude sensitivity to antenna lever arm (T = 0.78 sec, w = 450 deg/sec.

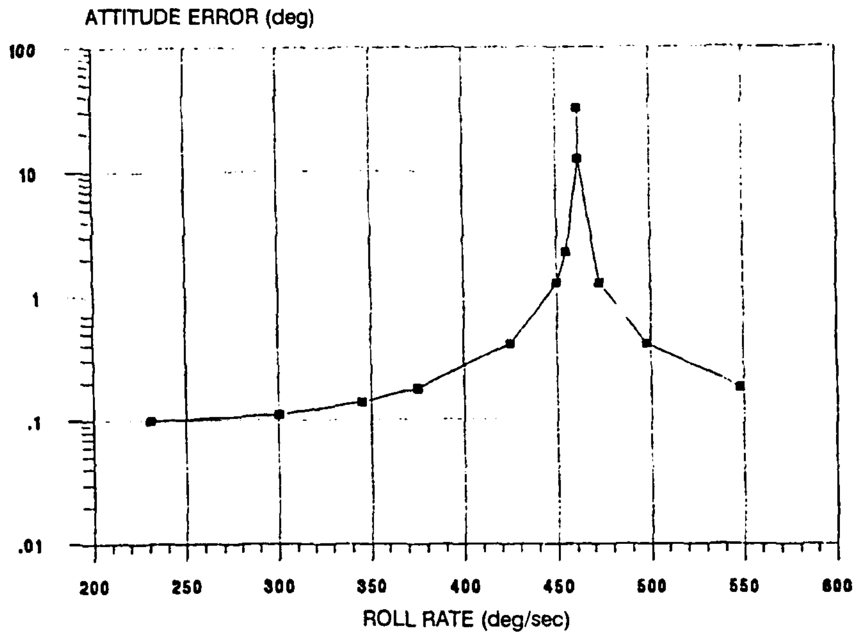


Figure 23. Final RSS attitude sensitivity to spin rate (T = 0.78 sec, L - 7 inch).

6.0 SUMMARY

A method was presented which permits estimation of attitude and attitude rate errors with an offset GPS antenna mounted on a platform undergoing attitude changes. The method exploits the attitude information inherently present in the pseudo range and the delta range measurements. The attitude information is primarily recovered through processing the delta range measurement with standard recursive estimation techniques such as the Kalman Filter algorithms of GPS/INS systems. A brief comparison to GPS interferometric techniques for attitude estimation was included.

Certain design considerations and performance sensitivities were identified and verified with covariance simulations. Specifically, attitude estimation performance is sensitive to the angle ωT angle swept during the DR integration, the length of the antenna lever arm, and satellite directions.

The method is ideally suited to spinning vehicles in exoatmospheric trajectories. In these applications recovery of attitude errors with conventional GPS/INS algorithms is not possible due to the decoupling between velocity and attitude errors. Certain benefits may be possible in terrestrial applications where rotary motion of the antenna platform is inherent, e.g., a helicopter with a GPS antenna mounted on the blade of the rotary wing.

In general, an integrated GPS/INS system is required for "good" performance. In that case, the implementation impact is minimum - a few terms in the observation matrix of the GPS/INS Kalman filter. A stand alone GPS receiver may suffice in certain applications such as benign or well known nominal angular dynamics. In that case, the implementation impact also includes constructing an attitude solution in near real time.

Although use of multiple offset antennas (and their associated receivers) has not been given specific treatment, the corresponding general observation matrix is simply obtained by incorporating a similar set of rows for each offset antenna/receiver combination. This may be accomplished whether the observation matrix is based on the use of pseudo-range or delta-range residuals or both.

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APPENDIX A

Given the error state vector, the nonlinear equation for the measurement process and its counterpart for the nominal trajectory

$$
\begin{aligned}
\delta\underline{x}(t) &= \underline{x}(t) - x_n(t) \\
\underline{z}(t_i) &= \underline{h}[\underline{x}(t_i), t_i] + \underline{v}(t_i) \\
\underline{z}_n(t_i) &= \underline{h}[\underline{x}_n(t_i), t_i].
\end{aligned} \tag{A1}
$$

The measurement residual is then

$$
\delta\underline{z}(t_i) = \underline{z}(t_i) - \underline{z}_n(t_i) = \underline{h}[\underline{x}(t_i), t_i] - \underline{h}[\underline{x}_n(t_i), t_i] + \underline{v}(t_i). \tag{A2}
$$

If we expand $\underline{h}[x(t_i), t_i]$ about $\underline{x}_n(t_i)$ using Taylor's Theorem

$$
\begin{aligned}
\mathbf{h}[\mathbf{x}(t_i), t_i] &= \mathbf{h}[\mathbf{x}_n(t_i), t_i] + \frac{\partial \mathbf{h}(\mathbf{x}, t_i)}{\partial \mathbf{x}}\Big|_{\mathbf{x}=\mathbf{x}_n(t_i)} [\mathbf{x}(t_i) - X_n(t_i)] \\
&\quad + o^2[\mathbf{x}(t_i) - \mathbf{x}_n(t_i)].
\end{aligned} \tag{A3}
$$

We then have

$$
\begin{aligned}
\underline{h}'[\delta\underline{x}(t_i), \underline{x}(t_i), t_i] &\triangleq \underline{h}[\underline{x}(t_i), t_i] - \underline{h}[\underline{x}_n(t_i), t_i] \\
&= \frac{\partial \underline{h}(\underline{x}, t_i)}{\partial \underline{x}}\Big|_{\underline{x}=\underline{x}_n(t_i)} \delta\underline{x} + o^2(\delta\underline{x})
\end{aligned} \tag{A4}
$$

and, therefore, by treating $\delta\underline{x}$ as an independent variable

$$
\frac{\partial \underline{h}'(\delta\underline{x}, \underline{x}, t_i)}{\partial(\delta\underline{x})} = \frac{\partial \underline{h}(\underline{x}, t_i)}{\partial \underline{x}}\Big|_{\underline{x}=\underline{x}_n(t_i)} + \tilde{o}(\delta\underline{x}). \tag{A5}
$$

Evaluating at $\delta\underline{x} = 0$

$$
\frac{\partial \underline{h}'(\delta\underline{x}, \underline{x}, t_i)}{\partial(\delta\underline{x})}\Big|_{\delta\underline{x}=\underline{0}} = \frac{\partial \underline{h}(\underline{x}, t_i)}{\partial \underline{x}}\Big|_{\underline{x}=\underline{x}_n(t_i)} = \underline{\underline{H}}[t_i; x_n(t_i)]. \tag{A6}
$$

If $\underline{h}[\underline{x}(t_i), t_i]$ has the more restrictive form

$$
\underline{h}[\underline{x}(t_i), t_i] = \begin{bmatrix} h_1[\underline{x}(t_i), t_i] \\ h_2[\underline{x}(t_i), t_i] \\ \vdots \\ h_m[\underline{x}(t_i), t_i] \end{bmatrix} \tag{A7}
$$

and, similarly,

$$\underline{h}'[\delta\underline{x}(t_i),\underline{x}(t_i),t_i] = \begin{bmatrix} h_1'[\delta\underline{x}(t_i),\underline{x}(t_i),t_i] \\ h_2'[\delta\underline{x}(t_i),\underline{x}(t_i),t_i] \\ \vdots \\ h_m'[\delta\underline{x}(t_i),\underline{x}(t_i),t_i] \end{bmatrix}, \tag{A8}$$

then we may deal only with a representative "row" for the above. In this event we have

$$\delta z_j(t_i) = h_j[\delta\underline{x}(t_i),\underline{x}(t_i),\underline{x},(t_i)] + \underline{v}_j(t_i)$$

$$\underline{h}_j^{T}{}''[t_i;\underline{x}_n(t_i)] = \frac{\partial h_j(\underline{x},t_i)}{\partial\underline{x}}\Big|_{x=x_n(t_i)} = \frac{\partial h_j'(\delta x,x,(t_i)}{\partial(\delta\underline{x})}\Big|_{\delta x=0}, \tag{A9}$$

where

$$\underline{\underline{H}}[t_i;\underline{x}_n(t_i)] = \begin{bmatrix} \underline{h}_1^{T}{}''[t_i;\underline{x}_n(t_i)] \\ \underline{h}_2^{T}{}''[t_i;\underline{x}_n(t_i)] \\ \vdots \\ \underline{h}_m^{T}{}''[t_i;\underline{x}_n(t_i)] \end{bmatrix}, \tag{A10}$$

and

$$\delta z(t_i) = \frac{\partial h_j''(\delta\underline{x},\underline{x},t_i)}{\partial(\delta\underline{x})}\Big|_{\delta\underline{x}=\underline{0}}\delta\underline{x}(t_i) + v_j(t_i) + \text{h.o.t..}$$

APPENDIX B

In this appendix, we show that

$$\frac{\partial \| \underline{V} + \underline{\underline{K}}\underline{u} \|}{\partial \underline{u}} \Big|_{\underline{u}=\underline{0}} = \frac{\underline{V}^T \underline{\underline{K}}}{\| \underline{V} \|} \text{ .} \tag{B1}$$

We note that $\frac{\partial}{\partial \underline{u}} = \left[\frac{\partial}{\partial u}, \frac{\partial}{\partial u_2}, \ldots \frac{\partial}{\partial u_k} \right]$ and functions as a post operator.

$$\begin{aligned} \frac{\partial \| \underline{V} + \underline{\underline{K}}\underline{u} \|}{\partial \underline{u}} &= \left[(\underline{V} + \underline{\underline{K}}\underline{u})^T \ (\underline{V} + \underline{\underline{K}}\underline{u}) \right]^{\frac{1}{2}} \frac{\partial}{\partial \underline{u}} \\ &= \frac{1}{2} \left[(\underline{V} + \underline{\underline{K}}\underline{u})^T (\underline{V} + \underline{\underline{K}}\underline{u} \right]^{-\frac{1}{2}} 2 \, (\underline{V} + \underline{\underline{K}}\underline{u})^T \underline{\underline{K}} \\ &= \frac{(\underline{V} + \underline{\underline{K}}\underline{u})^T \underline{\underline{K}}}{\| \underline{V} + \underline{\underline{K}}\underline{u} \|} \text{ .} \end{aligned} \tag{B2}$$

Therefore,

$$\frac{\partial \| \underline{V} + \underline{\underline{K}}\underline{u} \|}{\partial \underline{u}} \Big|_{\underline{u}=\underline{0}} = \frac{\underline{V}^T \underline{\underline{K}}}{\| \underline{V} \|} \text{ .} \tag{B3}$$

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In conventional integrated GPS/INS navigation systems, attitude errors are estimated through correlation to, primarily, velocity errors. This correlation develops due to the coupling of the attitude errors to the velocity errors via the presence of specific forces. Other GPS-based attitude measurement techniques typically employ multiple antennas and special receiver hardware.

An alternate approach to attitude estimation is proposed. This report exploits the attitude (and attitude rate) information inherently present in pseudo-range and, particularly, delta-range measurements obtained when a GPS antenna is offset from the GPS/INS reference point of a platform undergoing attitude changes, independent of the value of specific forces. A more general observation matrix that captures this information is derived. The attitude information is recovered with standard recursive estimation techniques such as the Kalman Filter algorithms typically used in these systems. The proposed approach is ideally suited for applications where the antenna is mounted on a spinning platform, such as a spinning vehicle in exoatmospheric, free-fall conditions. Simulation results are provided to illustrate the attitude estimation capability in a free-falling vehicle application. The technique developed here has the advantage of being readily implementable through minor software modifications (i.e., there are no special hardware requirements).

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